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The topological B-model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations

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Abstract: In the recent paper [1], it was argued that the open topological B-model whose target space is a complex (2|4)-dimensional mini-supertwistor space with D3- and D1-branes added corresponds to a super Yang-Mills theory in three dimensions. Without the D1-branes, this topological B-model is equivalent to a dimensionally reduced holomorphic Chern-Simons theory. Identifying the latter with a holomorphic BF-type theory, we describe a twistor correspondence between this theory and a supersymmetric Bogomolny model on $\mathbb{R}^3$. The connecting link in this correspondence is a partially holomorphic Chern-Simons theory on a Cauchy-Riemann supermanifold which is a real one-dimensional fibration over the mini-supertwistor space. Along the way of proving this twistor correspondence, we review the necessary basic geometric notions and construct action functionals for the involved theories. Furthermore, we discuss the geometric aspect of a recently proposed deformation of the mini-supertwistor space, which gives rise to mass terms in the supersymmetric Bogomolny equations. Eventually, we present solution generating techniques based on the developed twistorial description together with some examples and comment briefly on a twistor correspondence for super Yang-Mills theory in three dimensions.

Keywords: Chern-Simons Theories, Superspaces, Integrable Field Theories.

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1. Introduction and summary

Twistor string theory [2] is built upon the observation that the open topological B-model on the Calabi-Yau supermanifold given by the open subset $\mathcal{P}^{3|4} = \mathbb{C}P^{3|4} \backslash \mathbb{C}P^{1|4}$ of the supertwistor space $\mathbb{C}P^{3|4}$ with a stack of n D5-branes[1] is equivalent to holomorphic Chern-Simons (hCS) theory on the same space. This theory describes holomorphic structures on a rank n complex vector bundle $\mathcal{E}$ over $\mathcal{P}^{3|4}$ which are given by the (0,1) part $\mathcal{A}^{0,1}$ of a connection one-form $\mathcal{A}$ on $\mathcal{E}$. The components of $\mathcal{A}^{0,1}$ appear as the excitations of open strings ending on the D5-branes (i.e., they are zero modes of these strings). Furthermore, the spectrum of physical states contained in $\mathcal{A}^{0,1}$ is the same as that of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, but the interactions of both theories differ. In fact, by analyzing the linearized [2] and the full [3] field equations, it was shown that hCS theory on $\mathcal{P}^{3|4}$ is equivalent to the $\mathcal{N} = 4$ supersymmetric self-dual Yang-Mills (SDYM) theory on $\mathbb{R}^4$ introduced in [4], which can be considered as a truncation of the full $\mathcal{N} = 4$ SYM theory.

It was conjectured by Witten that the perturbative amplitudes of the full $\mathcal{N} = 4$ SYM theory are recovered by including D-instantons wrapping holomorphic curves in $\mathcal{P}^{3|4}$ into the topological B-model [2]. The presence of these D1-branes leads to additional fermionic states from zero modes of strings stretching between the D5- and the D1-branes. The scattering amplitudes are then computed in terms of currents constructed from these additional fields, which localize on the D1-branes, by integrating certain correlation functions over the moduli space of these D1-branes in $\mathcal{P}^{3|4}$. This proposal generalizes an earlier construction of maximally helicity-violating amplitudes by Nair [5]. Thus by incorporating D1-branes into the topological B-model, one can complement the $\mathcal{N} = 4$ SDYM theory to the full $\mathcal{N} = 4$ SYM theory, at least at tree-level. This conjecture has then been verified in several cases and it has been used in a number of papers for calculating field theory amplitudes by using methods inspired by string theory and twistor geometry. For a good account of the progress made in this area, see e.g. [6, 7] and references therein. For other aspects of twistor string theories discussed lately, see e.g. [8 – 14].

In a recent paper [1], a dimensional reduction of the above correspondence was considered: it was shown that scattering amplitudes of $\mathcal{N} = 4$ SYM theory which are localized on holomorphic curves in the supertwistor space $\mathcal{P}^{3|4}$ can be reduced to amplitudes of $\mathcal{N} = 8$ SYM theory in three dimensions which are localized on holomorphic curves in the supersymmetric extension $\mathcal{P}^{2|4}$ of the mini-twistor space $\mathcal{P}^2 := T^{1,0}\mathbb{C}P^1$. Note that the simplest of such curves in the mini-twistor space $\mathcal{P}^2$ is the Riemann sphere $\mathbb{C}P^1$ which coincides with

[1] These D5-branes are not quite space-filling and defined by the condition that all open string vertex operators do not depend on antiholomorphic Graßmann coordinates on $\mathcal{P}^{3|4}$.

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the *spectral curve* of the BPS SU(2) monopole.[2] The corresponding string theory after this reduction is the topological B-model on the mini-supertwistor space $\mathcal{P}^{2|4}$ with n not quite space-filling D3-branes (defined analogously to the D5-branes in the six-dimensional case) and additional D1-branes wrapping holomorphic cycles in $\mathcal{P}^{2|4}$. It is reasonable to assume that the latter correspond to monopoles and substitute the D-instantons in the case of the supertwistor space $\mathcal{P}^{3|4}$. In [1], also a twistor string theory corresponding to a certain massive SYM theory in three dimensions was described. The target space of the underlying topological B-model is a Calabi-Yau supermanifold obtained from the mini-supertwistor space $\mathcal{P}^{2|4}$ by deforming its complex structure along the fermionic directions.

The goal of this paper is to complement [1] by considering the open topological B-model on the Calabi-Yau supermanifold $\mathcal{P}^{2|4}$ in the presence of the D3-branes but without additional D1-branes. This model corresponds to a field theory on the mini-supertwistor space $\mathcal{P}^{2|4}$ obtained by a reduction of holomorphic Chern-Simons theory on the supertwistor space $\mathcal{P}^{3|4}$. We show that this field theory on $\mathcal{P}^{2|4}$ is a *holomorphic BF-type* (hBF) *theory*[3] which in turn is equivalent to a supersymmetric Bogomolny model. This model can be understood as a truncation of $\mathcal{N}=8$ SYM theory in three dimensions.

Recall that the open topological B-model on a Calabi-Yau (super)manifold Y with a stack of n (not quite) space-filling D-branes can be described in terms of $\operatorname{End}\mathcal{E}$-valued $(0,q)$-forms from the Dolbeault cohomology group $H^{0,q}_{\bar\partial}(Y,\operatorname{End}\mathcal{E})$, where $\mathcal{E}$ is a rank n vector bundle over Y. It was argued in [20, 2] that in three complex dimensions, the relevant field is the $(0,1)$ part $\mathcal{A}^{0,1}$ of a connection one-form $\mathcal{A}$ on the complex vector bundle $\mathcal{E}$. All the remaining fields are unphysical and only needed when quantizing the theory. This is also supported by an example presented in this paper: the B-model on the space $\mathcal{P}^{2|4}$ corresponds to a gauge theory, which contains besides $\mathcal{A}^{0,1}$ also an unphysical scalar field from $H^{0,0}_{\bar\partial}(\mathcal{P}^{2|4},\operatorname{End}\mathcal{E})$ as a Lagrange multiplier in the action functional. This theory describes again holomorphic structures on $\mathcal{E}$.

The action functional of hBF theory on $\mathcal{P}^{2|4}$ is not of Chern-Simons (CS) type, but one can introduce a CS type action on the correspondence space $\mathcal{F}^{5|8}\cong\mathbb{R}^{3|8}\times S^2$ which enters into the double fibration

$$\begin{tikzcd} & \mathcal{F}^{5|8} \arrow[dl] \arrow[dr] & \\ \mathcal{P}^{2|4} & & \mathbb{R}^{3|8} \end{tikzcd} \tag{1.1}$$

This diagram describes a correspondence between holomorphic projective lines in $\mathcal{P}^{2|4}$ and points in the euclidean superspace $\mathbb{R}^{3|8}$ obtained by a dimensional reduction of the superspace $\mathbb{R}^{4|8}$ along the x^4-axis. The correspondence space $\mathcal{F}^{5|8}$ admits a so-called *Cauchy-Riemann* (CR) *structure*, which can be considered as a generalization of a complex structure (see e.g. [21] for the purely bosonic case). After enlarging the integrable distribution defining this CR structure by one real direction to a distribution $\mathcal{T}$, one is led to the notion of $\mathcal{T}$-flat vector bundles over $\mathcal{F}^{5|8}$. These bundles take over the rôle of holomorphic vector

[2] Every static SU(2) monopole of charge k may be constructed from an algebraic curve in $\mathcal{P}^2$ [15], and an SU(n) monopole is defined by $n-1$ such holomorphic curves [16].

[3] These theories were introduced in [17] and considered e.g. in [18, 19].

bundles, and they can be defined by a $\mathcal{T}$-flat connection one-form $\mathcal{A}_{\mathcal{T}}$ [22]. The condition of $\mathcal{T}$-flatness of $\mathcal{A}_{\mathcal{T}}$ can be derived as the equations of motion of a theory we shall call *partially holomorphic Chern-Simons* (phCS) *theory*. This theory can be obtained by a dimensional reduction of hCS theory on the supertwistor space $\mathcal{P}^{3|4}$. We prove that there are one-to-one correspondences between equivalence classes of holomorphic vector bundles[4] over $\mathcal{P}^{2|4}$, equivalence classes of $\mathcal{T}$-flat vector bundles[4] over $\mathcal{F}^{5|8}$ and gauge equivalence classes of solutions to supersymmetric Bogomolny equations on $\mathbb{R}^3$. In other words, the moduli spaces of all three theories are bijective. Thus, we show that phCS theory is the connecting link between hBF theory on $\mathcal{P}^{2|4}$ and the supersymmetric Bogomolny model on $\mathbb{R}^3$:

$$
\begin{tikzcd}
& \text{phCS theory on } \mathcal{F}^{5|8} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{hBF theory on } \mathcal{P}^{2|4} \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{supersymmetric}\\ \text{Bogomolny model on } \mathbb{R}^3\end{array}
\end{tikzcd}
\tag{1.2}
$$

By a deformation of the complex structure on $\mathcal{P}^{2|4}$, which in turn induces a deformation of the CR structure on $\mathcal{F}^{5|8}$, we obtain a correspondence of the type (1.2) with additional mass terms for fermions and scalars in the supersymmetric Bogomolny equations.

The twistorial description of the supersymmetric Bogomolny equations on $\mathbb{R}^3$ has the nice feature that it yields novel methods for constructing explicit solutions. For simplicity, we restrict our discussion to solutions where only fields with helicity ± 1 and a Higgs field are nontrivial. The corresponding abelian configurations give rise to the Dirac monopole-antimonopole systems. For the non-abelian case, we present two ways of constructing solutions: First, by using a dressed version of the Penrose-Ward transform and second, by considering a nilpotent deformation of the holomorphic vector bundle corresponding to an arbitrary seed solution of the ordinary Bogomolny equations.

The organization of the paper is as follows. In section 2, we review the geometry of the (super)manifolds and the field theories involved in the $\mathcal{N}=4$ supertwistor correspondence. In particular, the equivalence of hCS theory on the supertwistor space $\mathcal{P}^{3|4} \cong \mathbb{R}^{4|8} \times S^2$ and $\mathcal{N}=4$ SDYM theory on $\mathbb{R}^4$ is recalled. Translations along the x^4-axis in $\mathbb{R}^4$ induce actions of a real and a complex one-parameter group on the space $\mathcal{P}^{3|4}$, which are described in section 3. Taking the quotient of $\mathcal{P}^{3|4}$ with respect to these groups yields the orbit spaces $\mathcal{F}^{5|8}$ and $\mathcal{P}^{2|4}$. In section 4, the partially holomorphic Chern-Simons theory (which is naturally defined on $\mathcal{F}^{5|8}$) is introduced and its equivalence to a supersymmetric Bogomolny model on $\mathbb{R}^3$ is proven. In section 5, we extend this equivalence to a holomorphic BF theory on the mini-supertwistor space $\mathcal{P}^{2|4}$, thus completing the picture (1.2). The deformations of the complex structure on the mini-supertwistor space $\mathcal{P}^{2|4}$ and of the CR structure on the space $\mathcal{F}^{5|8}$ which yield additional mass terms in the Bogomolny equations together with a detailed analysis of the geometric background are presented in section 6. Section 7 is concerned with the construction of explicit solutions to the supersymmetric Bogomolny equations: we describe two solution-generating algorithms and give some examples. Even-

[4]obeying certain triviality conditions

tually, we briefly comment on a twistor correspondence for the full $\mathcal{N}=8$ SYM theory in section 8. While appendices A and B cover some technical details, appendix C provides some remarks on the supertwistor correspondence for the case of signature $(++--)$.

2. Geometry of the $\mathcal{N}=4$ supertwistor space

2.1 Euclidean twistors in real and complex setting

General case. Let us consider a smooth oriented real four-manifold X with a metric g of signature $(++++)$ and the principal bundle $P(X, \mathrm{SO}(4))$ of orthonormal frames over X. The twistor space $\mathcal{Z}$ of X can be defined[5] as the associated bundle [23]

$$\mathcal{Z} := P(X, \mathrm{SO}(4)) \times_{\mathrm{SO}(4)} (\mathrm{SO}(4)/\,\mathrm{U}(2)) \tag{2.1}$$

with the canonical projection

$$\pi \,:\, \mathcal{Z} \;\to\; X\,. \tag{2.2}$$

The fibres of this bundle are two-spheres $S^2_x \cong \mathrm{SO}(4)/\,\mathrm{U}(2)$ which parametrize almost complex structures on the tangent spaces T_xX. As a real manifold, $\mathcal{Z}$ has dimension six.

Note that while a manifold X admits in general no almost complex structure, its twistor space $\mathcal{Z}$ can always be equipped with an almost complex structure $\mathcal{J}$ [23]. Furthermore, $\mathcal{J}$ is integrable if and only if the Weyl tensor of X is self-dual [24, 23]. Then $\mathcal{Z}$ is a complex three-manifold with an antiholomorphic involution τ (a real structure) which maps $\mathcal{J}$ to $-\mathcal{J}$ and the fibres of the bundle (2.2) over $x \in X$ are τ-invariant projective lines $\mathbb{C}P^1_x$, each of which has normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ in the complex manifold $\mathcal{Z}$.

The projective space $\mathbb{C}P^3$. It follows from (2.1) that the twistor space of the four-sphere S^4 endowed with the canonical conformally flat metric is the complex projective space $\mathbb{C}P^3$ [23],

$$\mathbb{C}P^3 \;\cong\; P(S^4, \mathrm{SO}(4)) \times_{\mathrm{SO}(4)} (\mathrm{SO}(4)/\,\mathrm{U}(2))\,. \tag{2.3}$$

In the following, we describe this space by the complex homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha})$ subject to the equivalence relation $(\omega^\alpha, \lambda_{\dot\alpha}) \sim (t\omega^\alpha, t\lambda_{\dot\alpha})$ for any $t \in \mathbb{C}^*$, where the spinor indices $\alpha, \beta, \ldots$ and $\dot\alpha, \dot\beta, \ldots$ run over $1,2$ and $\dot1, \dot2$, respectively. The real structure τ on $\mathbb{C}P^3$ is induced by the anti-linear transformations

$$\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \mapsto \begin{pmatrix}-\bar\omega^2\\ \bar\omega^1\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\lambda_{\dot1}\\ \lambda_{\dot2}\end{pmatrix} \mapsto \begin{pmatrix}-\bar\lambda_{\dot2}\\ \bar\lambda_{\dot1}\end{pmatrix}. \tag{2.4}$$

While there are no fixed points of τ in $\mathbb{C}P^3$, there are τ-invariant rational curves $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^3$.

[5]Further (equivalent) definitions of the twistor space $\mathcal{Z}$ can be found in appendix C.

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The twistor space $\mathcal{P}^3$ of $\mathbb{R}^4$. By definition (2.1), the twistor space of the euclidean space $\mathbb{R}^4$ is

$$\mathcal{Z} = P(\mathbb{R}^4, \mathrm{SO}(4)) \times_{\mathrm{SO}(4)} (\mathrm{SO}(4)/\,\mathrm{U}(2)) \;\cong\; \mathbb{R}^4 \times S^2\,. \tag{2.5}$$

Having in mind that $\mathbb{R}^4 \cong S^4\backslash\{\infty\}$, one can identify the twistor space $\mathcal{Z}$ of $\mathbb{R}^4$ with the complex three-manifold $\mathcal{P}^3 := \mathbb{C}P^3\backslash\mathbb{C}P^1_\infty$. Here, the point $\infty \in S^4$ corresponds to the projective line $\mathbb{C}P^1_\infty \subset \mathbb{C}P^3$. Note that we can choose to parametrize this sphere $\mathbb{C}P^1_\infty$ by the homogeneous coordinates $(\lambda_{\dot\alpha}) = (0,0)^T$ and $(\omega^\alpha) \neq (0,0)^T$. Thus, one can obtain $\mathcal{P}^3$ by taking the subset $(\lambda_{\dot\alpha}) \neq (0,0)^T$ on $\mathbb{C}P^3$ and a real structure τ on $\mathcal{P}^3$ is induced from the one on $\mathbb{C}P^3$. This space together with τ is diffeomorphic to the space (2.5),

$$\mathcal{P}^3 = \mathbb{C}P^3\backslash\mathbb{C}P^1_\infty \;\cong\; \mathbb{R}^4 \times S^2 \tag{2.6}$$

and therefore $\mathcal{P}^3$ is the twistor space of $\mathbb{R}^4$ with the canonical projection

$$\pi \;:\; \mathcal{P}^3 \;\to\; \mathbb{R}^4\,. \tag{2.7}$$

We can cover $\mathcal{P}^3$ by two patches $\mathcal{U}_+$ and $\mathcal{U}_-$ for which $\lambda_{\dot 1} \neq 0$ and $\lambda_{\dot 2} \neq 0$, respectively, and introduce the coordinates

$$\begin{aligned} z_+^\alpha &= \frac{\omega^\alpha}{\lambda_{\dot 1}}\,, \qquad z_+^3 = \frac{\lambda_{\dot 2}}{\lambda_{\dot 1}} =: \lambda_+ \qquad \text{on} \qquad \mathcal{U}_+\,,\\ z_-^\alpha &= \frac{\omega^\alpha}{\lambda_{\dot 2}}\,, \qquad z_-^3 = \frac{\lambda_{\dot 1}}{\lambda_{\dot 2}} =: \lambda_- \qquad \text{on} \qquad \mathcal{U}_-\,, \end{aligned} \tag{2.8}$$

which are related on $\mathcal{U}_+ \cap \mathcal{U}_-$ by the equations

$$z_+^\alpha = \frac{1}{z_-^3}\, z_-^\alpha \qquad \text{and} \qquad z_+^3 = \frac{1}{z_-^3}\,. \tag{2.9}$$

From this, it follows that $\mathcal{P}^3$ coincides with the total space of the rank 2 holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over the Riemann sphere $\mathbb{C}P^1$, i.e.

$$\mathcal{P}^3 \;\to\; \mathbb{C}P^1 \quad \text{with} \quad \mathcal{P}^3 = \mathcal{O}(1) \oplus \mathcal{O}(1)\,. \tag{2.10}$$

The base manifold $\mathbb{C}P^1$ of this fibre bundle is covered by two patches $U_\pm = \mathcal{U}_\pm \cap \mathbb{C}P^1$ with affine coordinates $\lambda_\pm$.

The real structure τ on $\mathcal{P}^3$ induced by the transformations (2.4) acts on the coordinates (2.8) as follows:

$$\tau(z_\pm^1, z_\pm^2, z_\pm^3) = \left(\pm\frac{\bar z_\pm^2}{\bar z_\pm^3}, \mp\frac{\bar z_\pm^1}{\bar z_\pm^3}, -\frac{1}{\bar z_\pm^3}\right)\,. \tag{2.11}$$

It is not difficult to see that (analogously to the case of $\mathbb{C}P^3$) τ has no fixed points in $\mathcal{P}^3$ but leaves invariant projective lines $\mathbb{C}P^1$ joining the points p and $\tau(p)$ for any $p \in \mathcal{P}^3$. For two other possible real structures on $\mathcal{P}^3$, see e.g. [3].

Incidence relations. Global holomorphic sections of the bundle (2.10) are locally polynomials of degree one in $\lambda_\pm$. Introducing the spinorial notation

$$(\lambda^+_{\dot\alpha}) := \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \qquad \text{and} \qquad (\lambda^-_{\dot\alpha}) := \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} , \tag{2.12}$$

one can parametrize these sections by the moduli $x = (x^{\alpha\dot\alpha}) \in \mathbb{C}^4$ as

$$z^\alpha_\pm = x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha} \tag{2.13}$$

over the patches $U_\pm$. These sections describe a holomorphic embedding of rational curves[6] $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ for fixed $x \in \mathbb{C}^4$. On the other hand, for each point $p = (z^\alpha_\pm, \lambda^\pm_{\dot\alpha}) \in \mathcal{P}^3$, the incidence relations (2.13) define a null (anti-self-dual) two-plane (β-plane) in $\mathbb{C}^4$. The correspondences

$$\begin{aligned} \{\,\text{projective lines } \mathbb{C}P^1_x \text{ in } \mathcal{P}^3\} &\longleftrightarrow \{\,\text{points } x \text{ in } \mathbb{C}^4\}\,, \\ \{\,\text{points } p \text{ in } \mathcal{P}^3\} &\longleftrightarrow \{\,\beta\text{-planes } \mathbb{C}^2_p \text{ in } \mathbb{C}^4\} \end{aligned} \tag{2.14}$$

between subspaces in $\mathcal{P}^3$ and $\mathbb{C}^4$ can be described by a double fibration (see e.g. [3] and references therein). Those curves $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ which are invariant under the involution τ defined in (2.11) are parametrized by moduli $x \in \mathbb{C}^4$ satisfying the equations

$$\tau \begin{pmatrix} x^{1\dot1} & x^{1\dot2} \\ x^{2\dot1} & x^{2\dot2} \end{pmatrix} = \begin{pmatrix} \bar x^{2\dot2} & -\bar x^{2\dot1} \\ -\bar x^{1\dot2} & \bar x^{1\dot1} \end{pmatrix} = \begin{pmatrix} x^{1\dot1} & x^{1\dot2} \\ x^{2\dot1} & x^{2\dot2} \end{pmatrix} , \tag{2.15}$$

and therefore we can introduce real coordinates $(x^\mu) \in \mathbb{R}^4$ with $\mu = 1, \dots, 4$ by[7]

$$x^{2\dot2} = \bar x^{1\dot1} =: -\mathrm{i}(x^1 - \mathrm{i}x^2) \qquad \text{and} \qquad x^{2\dot1} \;=\; -\bar x^{1\dot2} =: -\mathrm{i}(x^3 - \mathrm{i}x^4)\,. \tag{2.16}$$

These are coordinates on the base of the fibration $\mathcal{P}^3 \to \mathbb{R}^4$, which parametrize τ-real holomorphic curves $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$, and come naturally with the euclidean metric

$$\mathrm{d}s^2 = \det(\mathrm{d}x^{\alpha\dot\alpha}) = \delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{2.17}$$

Other real structures on $\mathcal{P}^3$ give rise to a metric with signature $(++--)$ on $\mathbb{R}^4$.

2.2 The twistor space of $\mathbb{R}^4$ as a direct product of complex manifolds

Coordinate transformations. Recall that the twistor space of $\mathbb{R}^4$ can be considered both as the smooth manifold $\mathbb{R}^4 \times S^2$ and as the complex manifold $\mathcal{P}^3$ because there is a diffeomorphism between them. Switching to the coordinates $x^{\alpha\dot1}$ from (2.16) and to $\lambda_\pm = \mathrm{Re}\,\lambda_\pm + \mathrm{i}\,\mathrm{Im}\,\lambda_\pm$, we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and S^2 with $\mathbb{C}P^1$, respectively. Thus, we have further diffeomorphisms

$$\mathbb{R}^4 \times S^2 \;\cong\; \mathbb{C}^2 \times \mathbb{C}P^1 \;\cong\; \mathcal{P}^3\,. \tag{2.18}$$

[6]By the Kodaira theorem [25], the complex dimensions of the moduli space parametrizing a family of rational curves embedded holomorphically into $\mathcal{P}^3$ is $\dim_{\mathbb{C}} H^0(\mathbb{C}P^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 4$ and there are no obstructions to these deformations since $H^1(\mathbb{C}P^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 0$.

[7]Note that our choice of relations between $x^{\alpha\dot\alpha}$ and x^μ differs from that of [3].

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The latter is defined by (2.13) and (2.15), and its inverse reads explicitly as

$$x^{1\dot{1}} = \frac{z_+^1 + z_+^3 \bar{z}_+^2}{1 + z_+^3 \bar{z}_+^3} = \frac{\bar{z}_-^3 z_-^1 + \bar{z}_-^2}{1 + z_-^3 \bar{z}_-^3}\,, \qquad x^{1\dot{2}} = -\frac{\bar{z}_+^2 - \bar{z}_+^3 z_+^1}{1 + z_+^3 \bar{z}_+^3} = -\frac{z_-^3 \bar{z}_-^2 - z_-^1}{1 + z_-^3 \bar{z}_-^3}\,,$$
$$\lambda_\pm = z_\pm^3\,. \tag{2.19}$$

Note that while the complex manifolds $\mathcal{P}^3$ and $\mathbb{C}^2 \times \mathbb{C}P^1$ are diffeomorphic, they are not biholomorphic as their complex structures obviously differ.

Vector fields. On the complex manifold $\mathcal{P}^3$, we have the natural basis $\{\frac{\partial}{\partial \bar{z}_\pm^\alpha}, \frac{\partial}{\partial \bar{z}_\pm^3}\}$ for the space of antiholomorphic vector fields. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$, we find

$$\frac{\partial}{\partial \bar{z}_+^\alpha} = \bar{z}_-^3 \frac{\partial}{\partial \bar{z}_-^\alpha} \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_+^3} = -(\bar{z}_-^3)^2 \frac{\partial}{\partial \bar{z}_-^3} - \bar{z}_-^3 \bar{z}_-^\alpha \frac{\partial}{\partial \bar{z}_-^\alpha}\,. \tag{2.20}$$

Using formulæ (2.19), we can express these vector fields in terms of the coordinates $(x^{\alpha\dot{1}}, \lambda_\pm)$ and their complex conjugates according to

$$\frac{\partial}{\partial \bar{z}_\pm^1} = -\gamma_\pm \lambda_\pm^{\dot{\alpha}} \frac{\partial}{\partial x^{2\dot{\alpha}}} =: -\gamma_\pm \bar{V}_2^\pm\,, \qquad \frac{\partial}{\partial \bar{z}_\pm^2} = \gamma_\pm \lambda_\pm^{\dot{\alpha}} \frac{\partial}{\partial x^{1\dot{\alpha}}} =: \gamma_\pm \bar{V}_1^\pm\,,$$
$$\frac{\partial}{\partial \bar{z}_+^3} = \frac{\partial}{\partial \bar{\lambda}_+} - \gamma_+ x^{\alpha\dot{1}} \bar{V}_\alpha^+\,, \qquad \frac{\partial}{\partial \bar{z}_-^3} = \frac{\partial}{\partial \bar{\lambda}_-} - \gamma_- x^{\alpha\dot{2}} \bar{V}_\alpha^-\,, \tag{2.21}$$

where we have used

$$\lambda_\pm^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\beta}}^\pm \qquad \text{with} \qquad \varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}} = 1 \qquad \text{and} \qquad \gamma_\pm := \frac{1}{1 + \lambda_\pm \bar{\lambda}_\pm} = \frac{1}{\hat{\lambda}_\pm^{\dot{\alpha}} \lambda_{\dot{\alpha}}^\pm} \tag{2.22}$$

together with the convention $\varepsilon_{\dot{1}\dot{2}} = -\varepsilon_{\dot{2}\dot{1}} = -1$, which implies $\varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}$. Thus, the vector fields

$$\bar{V}_\alpha^\pm = \lambda_\pm^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \qquad \text{and} \qquad \bar{V}_3^\pm := \frac{\partial}{\partial \bar{\lambda}_\pm} \tag{2.23}$$

form a basis of vector fields of type $(0,1)$ on $\mathcal{U}_\pm \subset \mathcal{P}^3$ in the coordinates $(x^{\alpha\dot{\alpha}}, \lambda_\pm, \bar{\lambda}_\pm)$.

Forms. It is easy to check that the basis of (0,1)-forms on $\mathcal{U}_\pm$, which are dual to the vector fields (2.23), is given by

$$\bar{E}_\pm^\alpha = -\gamma_\pm \hat{\lambda}_{\dot{\alpha}}^\pm \mathrm{d}x^{\alpha\dot{\alpha}} \qquad \text{and} \qquad \bar{E}_\pm^3 = \mathrm{d}\bar{\lambda}_\pm\,, \tag{2.24}$$

where

$$(\hat{\lambda}_{\dot{\alpha}}^+) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\lambda}_+ \end{pmatrix} = \begin{pmatrix} -\bar{\lambda}_+ \\ 1 \end{pmatrix},$$
$$(\hat{\lambda}_{\dot{\alpha}}^-) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\lambda}_- \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ \bar{\lambda}_- \end{pmatrix}. \tag{2.25}$$

One can easily verify that

$$\bar{\partial}|_{\mathcal{U}_\pm} = \mathrm{d}\bar{z}_\pm^a \frac{\partial}{\partial \bar{z}_\pm^a} = \bar{E}_\pm^a \bar{V}_a^\pm \qquad \text{for} \qquad a = 1,2,3\,, \tag{2.26}$$

and $\bar{E}_+^a \bar{V}_a^+ = \bar{E}_-^a \bar{V}_a^-$ on $\mathcal{U}_+ \cap \mathcal{U}_-$. More details on twistor theory can be found in the books [26 – 29].

2.3 Supertwistor spaces as complex supermanifolds

The supermanifolds $\mathbb{C}P^{3|4}$ and $\mathcal{P}^{3|4}$. An extension of the twistor space $\mathbb{C}P^3$ to a Calabi-Yau supermanifold is the space $\mathbb{C}P^{3|4}$ which is described by homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha}, \eta_i) \in \mathbb{C}^{4|4}\backslash\{0\}$ subject to the identification $(\omega^\alpha, \lambda_{\dot\alpha}, \eta_i) \sim (t\omega^\alpha, t\lambda_{\dot\alpha}, t\eta_i)$ for any $t \in \mathbb{C}^*$ [2]. Here, $(\omega^\alpha, \lambda_{\dot\alpha})$ are the homogeneous coordinates on the body $\mathbb{C}P^3$ and η_i with $i = 1, \ldots, 4$ are complex Graßmann variables.

Similarly to the bosonic case, we introduce the space $\mathcal{P}^{3|4} := \mathbb{C}P^{3|4}\backslash\mathbb{C}P^{1|4}_\infty$ by demanding that the $\lambda_{\dot\alpha}$ are not simultaneously zero. This space is an open subset of $\mathbb{C}P^{3|4}$ covered by two patches $\hat{\mathcal{U}}_+$ and $\hat{\mathcal{U}}_-$ with bosonic coordinates (2.8) and fermionic coordinates

$$\eta_i^+ = \frac{\eta_i}{\lambda_{\dot 1}} \qquad \text{on} \qquad \hat{\mathcal{U}}_+ \qquad \text{and} \qquad \eta_i^- = \frac{\eta_i}{\lambda_{\dot 2}} \qquad \text{on} \qquad \hat{\mathcal{U}}_- \,. \tag{2.27}$$

The latter are related by $\eta_i^+ = (z_-^3)^{-1}\eta_i^-$ on the intersection $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. From this, it becomes clear that $\mathcal{P}^{3|4}$ is the following holomorphic vector bundle[8] over $\mathbb{C}P^1$:

$$\mathcal{P}^{3|4} \;\to\; \mathbb{C}P^1 \qquad \text{with} \qquad \mathcal{P}^{3|4} = \mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^4 \,. \tag{2.28}$$

The fibres over $\lambda \in \mathbb{C}P^{1|0} \equiv \mathbb{C}P^1$ are the superspaces $\mathbb{C}^{2|4}_\lambda$. In the following, we will refer to $\mathcal{P}^{3|4}$ as the ($\mathcal{N} = 4$) *supertwistor space*.

Incidence relations. Holomorphic sections of the bundle (2.28) are polynomials of degree one in $\lambda_\pm$. They are defined by the equations

$$z_\pm^\alpha = x_R^{\alpha\dot\alpha}\,\lambda_{\dot\alpha}^\pm \qquad \text{and} \qquad \eta_i^\pm = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm \qquad \text{on} \qquad \hat{\mathcal{U}}_\pm \tag{2.29}$$

and parametrized by the moduli $(x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \in \mathbb{C}^{4|8}$. The latter space is called the $\mathcal{N} = 4$ (complex) anti-chiral superspace. In the following, we omit the subscript R for brevity.

Equations (2.29) define a curve $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|4}$ for fixed $(x, \eta) = (x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \in \mathbb{C}^{4|8}$ and a null β-superplane of complex dimension $(2|4)$ in $\mathbb{C}^{4|8}$ for fixed $p = (z_\pm^\alpha, \lambda_\pm, \eta_i^\pm) \in \mathcal{P}^{3|4}$. Thus, the incidence relations (2.29) yield the correspondences

$$\begin{aligned} \{\,\text{projective lines } \mathbb{C}P^1_{x,\eta} \text{ in } \mathcal{P}^{3|4}\} &\longleftrightarrow \{\,\text{points } (x,\eta) \text{ in } \mathbb{C}^{4|8}\}\,, \\ \{\,\text{points } p \text{ in } \mathcal{P}^{3|4}\} &\longleftrightarrow \{\,\beta\text{-superplanes } \mathbb{C}^{2|4}_p \text{ in } \mathbb{C}^{4|8}\}\,, \end{aligned} \tag{2.30}$$

which can be described by a double fibration (see e.g. [3]).

2.4 Real structure on the $\mathcal{N} = 4$ supertwistor space

Reality conditions. The antiholomorphic involution $\tau : \mathcal{P}^3 \to \mathcal{P}^3$ given by (2.11) can be extended to an antiholomorphic involution $\tau : \mathcal{P}^{3|4} \to \mathcal{P}^{3|4}$ by defining

$$\tau(\eta_i^\pm) = \pm\frac{1}{\bar z_\pm^3} T_i{}^j \bar\eta_j^\pm \qquad \text{with} \qquad (T_i{}^j) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.31}$$

[8] The operator Π inverts the parity of the fibre coordinates of a vector bundle.

for the fermionic coordinates $\eta_i^\pm$ on $\mathcal{P}^{3|4}$. On the moduli space $\mathbb{C}^{4|8}$, this corresponds to the involution (2.15) for the bosonic coordinates $x^{\alpha\dot\alpha}$ and to

$$\tau(\eta_i^{\dot\alpha}) = \varepsilon^{\dot\alpha\dot\beta} T_i{}^j \bar\eta_j^{\dot\beta} \quad\Leftrightarrow\quad \tau \begin{pmatrix} \eta_1^{\dot 1} & \eta_2^{\dot 1} & \eta_3^{\dot 1} & \eta_4^{\dot 1} \\ \eta_1^{\dot 2} & \eta_2^{\dot 2} & \eta_3^{\dot 2} & \eta_4^{\dot 2} \end{pmatrix} = \begin{pmatrix} -\bar\eta_2^{\dot 2} & \bar\eta_1^{\dot 2} & -\bar\eta_4^{\dot 2} & \bar\eta_3^{\dot 2} \\ \bar\eta_2^{\dot 1} & -\bar\eta_1^{\dot 1} & \bar\eta_4^{\dot 1} & -\bar\eta_3^{\dot 1} \end{pmatrix} \tag{2.32}$$

for the fermionic coordinates $\eta_i^{\dot\alpha}$. From (2.32), one can directly read off the reality conditions

$$\tau(\eta_i^{\dot\alpha}) = \eta_i^{\dot\alpha} \qquad\Leftrightarrow\qquad \eta_i^{\dot 2} \;=\; -T_i{}^j \bar\eta_j^{\dot 1}\,, \tag{2.33}$$

which we impose on the Graßmann variables $\eta_i^{\dot\alpha}$. These reality conditions together with (2.16) define the $\mathcal{N}=4$ (anti-chiral) superspace $\mathbb{R}^{4|8} \cong \mathbb{C}^{2|4}$. Thus, $\mathcal{P}^{3|4}$ is the supertwistor space for the euclidean superspace $\mathbb{R}^{4|8}$.

Coordinate transformations. Note that formulæ (2.29) with $x^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$ obeying the reality conditions (2.16) and (2.33) define the diffeomorphisms

$$\mathbb{R}^{4|8} \times S^2 \;\cong\; \mathbb{C}^{2|4} \times \mathbb{C}P^1 \;\cong\; \mathcal{P}^{3|4}\,. \tag{2.34}$$

The map from $\mathcal{P}^{3|4}$ to the space $\mathbb{C}^{2|4} \times \mathbb{C}P^1$ with complex coordinates $(x^{\alpha\dot 1}, \eta_i^{\dot 1}, \lambda_\pm)$ is given by (2.19) and

$$\begin{aligned} \eta_1^{\dot 1} &= \frac{\eta_1^+ - z_+^3 \bar\eta_2^+}{1 + z_+^3 \bar z_+^3} = \frac{\bar z_-^3 \eta_1^- - \bar\eta_2^-}{1 + z_-^3 \bar z_-^3}\,, \qquad & \eta_2^{\dot 1} &= \frac{\eta_2^+ + z_+^3 \bar\eta_1^+}{1 + z_+^3 \bar z_+^3} = \frac{\bar z_-^3 \eta_2^- + \bar\eta_1^-}{1 + z_-^3 \bar z_-^3}\,, \\ \eta_3^{\dot 1} &= \frac{\eta_3^+ - z_+^3 \bar\eta_4^+}{1 + z_+^3 \bar z_+^3} = \frac{\bar z_-^3 \eta_3^- - \bar\eta_4^-}{1 + z_-^3 \bar z_-^3}\,, \qquad & \eta_4^{\dot 1} &= \frac{\eta_4^+ + z_+^3 \bar\eta_3^+}{1 + z_+^3 \bar z_+^3} = \frac{\bar z_-^3 \eta_4^- + \bar\eta_3^-}{1 + z_-^3 \bar z_-^3}\,, \end{aligned} \tag{2.35}$$

together with (2.33). The formulæ (2.19) and (2.35) define also a (smooth) projection

$$\mathcal{P}^{3|4} \;\to\; \mathbb{R}^{4|8}\,. \tag{2.36}$$

Odd vector fields and forms. The odd antiholomorphic vector fields $\frac{\partial}{\partial \bar\eta_i^\pm}$ on $\mathcal{P}^{3|4}$ can be expressed due to (2.33) and (2.35) in terms of coordinates on $\mathbb{C}^{2|4} \times \mathbb{C}P^1$ as follows:

$$\begin{aligned} \frac{\partial}{\partial \bar\eta_1^\pm} &= \gamma_\pm \lambda_\pm^{\dot\alpha} \frac{\partial}{\partial \eta_2^{\dot\alpha}} \;=:\; \gamma_\pm \bar V_\pm^2\,, \qquad & \frac{\partial}{\partial \bar\eta_2^\pm} &= -\gamma_\pm \lambda_\pm^{\dot\alpha} \frac{\partial}{\partial \eta_1^{\dot\alpha}} \;=:\; -\gamma_\pm \bar V_\pm^1\,, \\ \frac{\partial}{\partial \bar\eta_3^\pm} &= \gamma_\pm \lambda_\pm^{\dot\alpha} \frac{\partial}{\partial \eta_4^{\dot\alpha}} \;=:\; \gamma_\pm \bar V_\pm^4\,, \qquad & \frac{\partial}{\partial \bar\eta_4^\pm} &= -\gamma_\pm \lambda_\pm^{\dot\alpha} \frac{\partial}{\partial \eta_3^{\dot\alpha}} \;=:\; -\gamma_\pm \bar V_\pm^3\,, \end{aligned} \tag{2.37}$$

or

$$\frac{\partial}{\partial \bar\eta_i^\pm} = \gamma_\pm T_j{}^i \bar V_\pm^j \tag{2.38}$$

for short. Therefore, the odd vector fields

$$\bar V_\pm^i = \lambda_\pm^{\dot\alpha} \frac{\partial}{\partial \eta_i^{\dot\alpha}} \tag{2.39}$$

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complement the vector fields (2.23) to a basis of vector fields of type (0,1) on $\hat{\mathcal{U}}_\pm \subset \mathcal{P}^{3|4}$ in the coordinates $(x^{\alpha\dot\alpha}, \lambda_\pm, \bar\lambda_\pm, \eta_i^{\dot\alpha})$. The basis of odd (0,1)-forms dual to the vector fields (2.39) is given by

$$\bar{E}_i^\pm = -\gamma_\pm \hat\lambda^\pm_{\dot\alpha} \mathrm{d}\eta_i^{\dot\alpha}\,. \tag{2.40}$$

Note that on the supermanifold $\mathcal{P}^{3|4}$, the transformations for $\frac{\partial}{\partial \bar{z}^3_\pm}$ in (2.21) are changed to

$$\frac{\partial}{\partial \bar{z}^3_+} = \frac{\partial}{\partial \bar\lambda_+} - \gamma_+ x^{\alpha\dot 1} \bar{V}^+_\alpha - \gamma_+ \eta_i^{\dot 1} \bar{V}^i_+ \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}^3_-} = \frac{\partial}{\partial \bar\lambda_-} - \gamma_- x^{\alpha\dot 2} \bar{V}^-_\alpha + \gamma_- \eta_i^{\dot 2} \bar{V}^i_-\,, \tag{2.41}$$

and we obtain

$$\bar\partial|_{\hat{\mathcal{U}}_\pm} = \mathrm{d}\bar{z}^a_\pm \frac{\partial}{\partial \bar{z}^a_\pm} + \mathrm{d}\bar\eta_i^\pm \frac{\partial}{\partial \bar\eta_i^\pm} = \bar{E}^a_\pm \bar{V}^\pm_a + \bar{E}^\pm_i \bar{V}^i_\pm\,. \tag{2.42}$$

For a discussion of other real structures on $\mathcal{P}^{3|4}$ related with signature $(++--)$, see e.g. [3].

Holomorphic integral form. Let us furthermore introduce the (nowhere vanishing) holomorphic volume element

$$\hat\Omega|_{\hat{\mathcal{U}}_\pm} := \pm \mathrm{d}z^1_\pm \wedge \mathrm{d}z^2_\pm \wedge \mathrm{d}z^3_\pm \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm \tag{2.43}$$

on $\mathcal{P}^{3|4}$. This holomorphic volume element exists since the berezinian of $T^{1,0}\mathcal{P}^{3|4}$ is a trivial bundle, and this implies that $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold. Note, however, that $\hat\Omega$ is not a differential form in the Graßmann coordinates, since Graßmann differential forms (as the ones used e.g. in (2.40)) are dual to Graßmann vector fields and thus transform contragrediently to them. Berezin integration is, however, equivalent to differentiation, and thus a volume element has to transform as a product of Graßmann vector fields, i.e. with the inverse of the jacobian. Such forms are called *integral forms* and for short, we will call $\hat\Omega$ a *holomorphic volume form*, similarly to the usual nomenclature for Calabi-Yau manifolds.

2.5 Holomorphic Chern-Simons theory on $\mathcal{P}^{3|4}$

Recall that the open topological B-model on a complex three-dimensional Calabi-Yau manifold with a stack of n D5-branes is equivalent to holomorphic Chern-Simons theory and describes holomorphic structures on a rank n vector bundle over the same space. In the following, we will study this setting on the supertwistor space $\mathcal{P}^{3|4}$.

Equations of motion. Consider a trivial rank n complex vector bundle $\mathcal{E}$ over $\mathcal{P}^{3|4}$ and a connection one-form $\mathcal{A}$ on $\mathcal{E}$. We can use the holomorphic volume form (2.43) to write down an action for holomorphic Chern-Simons theory on $\mathcal{P}^{3|4}$,

$$S_{\mathrm{hCS}} = \int_{\mathscr{Y}} \hat\Omega \wedge \mathrm{tr}\left(\mathcal{A}^{0,1} \wedge \bar\partial \mathcal{A}^{0,1} + \frac{2}{3} \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1}\right), \tag{2.44}$$

where $\mathscr{Y}$ is the subspace of $\mathcal{P}^{3|4}$ for which $\bar\eta_i^\pm = 0$, $\bar\partial$ is the antiholomorphic part of the exterior derivative on $\mathcal{P}^{3|4}$ and $\mathcal{A}^{0,1}$ is the $(0,1)$ component of $\mathcal{A}$ which we assume to satisfy[9] $\bar{V}^i_\pm(\bar{V}^\pm_a \lrcorner \mathcal{A}^{0,1}) = 0$ and $\bar{V}^i_\pm \lrcorner \mathcal{A}^{0,1} = 0$. The equations of motion of this theory are

[9]Here, "$\lrcorner$" denotes the interior product of vector fields with differential forms.

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readily derived to be

$$\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0\,. \tag{2.45}$$

Equivalence to $\mathcal{N} = 4$ SDYM theory. By linearizing (2.45) around the trivial solution $\mathcal{A}^{0,1} = 0$, Witten has shown [2] that the equations (2.45) are equivalent to the field equations of $\mathcal{N} = 4$ self-dual Yang-Mills (SDYM) theory[10] on $\mathbb{R}^4$. On the full nonlinear level, this equivalence was demonstrated in [3]. The $\mathcal{N} = 4$ SDYM equations on the euclidean space $\mathbb{R}^4$ take the form

$$\begin{aligned}
f_{\dot\alpha\dot\beta} &= 0\,,\\
\varepsilon^{\alpha\beta} D_{\alpha\dot\alpha}\chi^i_\beta &= 0\,,\\
\Box\phi^{ij} &= -\varepsilon^{\alpha\beta}\{\chi^i_\alpha, \chi^j_\beta\}\,,\\
\varepsilon^{\dot\alpha\dot\beta} D_{\alpha\dot\alpha}\tilde\chi_{i\dot\beta} &= 2[\phi_{ij}, \chi^j_\alpha]\,,\\
\varepsilon^{\dot\alpha\dot\beta} D_{\alpha\dot\alpha} G_{\dot\beta\dot\gamma} &= \{\chi^i_\alpha, \tilde\chi_{i\dot\gamma}\} - \frac{1}{2}[\phi_{ij}, D_{\alpha\dot\gamma}\phi^{ij}]\,,
\end{aligned} \tag{2.46}$$

where $f_{\dot\alpha\dot\beta} := -\frac{1}{2}\varepsilon^{\alpha\beta}F_{\alpha\dot\alpha\beta\dot\beta}$ denotes the anti-self-dual part of the curvature $F_{\alpha\dot\alpha\beta\dot\beta} = [D_{\alpha\dot\alpha}, D_{\beta\dot\beta}]$. The fields in the $\mathcal{N} = 4$ supermultiplet $(f_{\alpha\beta}, \chi^i_\alpha, \phi^{ij}, \tilde\chi_{i\dot\alpha}, G_{\dot\alpha\dot\beta})$ carry the helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$, and $A_{\alpha\dot\alpha}$ in $D_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + A_{\alpha\dot\alpha}$ are the components of a (self-dual) gauge potential. Furthermore, we have introduced the abbreviation $\phi_{ij} := \frac{1}{2!}\varepsilon_{ijkl}\phi^{kl}$ as well as the operator $\Box := \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}D_{\alpha\dot\alpha}D_{\beta\dot\beta}$. Note that all the fields live in the adjoint representation of the gauge group. For proving this equivalence via a twistor correspondence, one considers only those gauge potentials $\mathcal{A}^{0,1}$ for which the component $\partial_{\bar\lambda_\pm} \lrcorner \mathcal{A}^{0,1}$ can be gauged away [3]. This means, that one works with a subset[11] in the set of all solutions of hCS theory on $\mathcal{P}^{3|4}$. In the following, we always imply this restriction when speaking about a twistor correspondence.

Čech description. Note that solutions $\mathcal{A}^{0,1}$ to (2.45) define holomorphic structures $\bar\partial_\mathcal{A} = \bar\partial + \mathcal{A}^{0,1}$ on $\mathcal{E}$ and thus $(\mathcal{E}, \bar\partial_\mathcal{A})$ is a holomorphic vector bundle (with trivial transition functions) in the Dolbeault description. To switch to the Čech description, we consider the restrictions of $\mathcal{A}^{0,1}$ to the patches $\hat{\mathcal{U}}_\pm$. Since $\mathcal{A}^{0,1}$ is flat due to (2.45), we can write it locally as a pure gauge configuration,

$$\mathcal{A}^{0,1}|_{\hat{\mathcal{U}}_+} = \psi_+\bar\partial\psi_+^{-1} \qquad \text{and} \qquad \mathcal{A}^{0,1}|_{\hat{\mathcal{U}}_-} = \psi_-\bar\partial\psi_-^{-1}\,, \tag{2.47}$$

where the $\psi_\pm$ are $\mathrm{GL}(n,\mathbb{C})$-valued functions on $\hat{\mathcal{U}}_\pm$ such that $\bar V^i_\pm\psi_\pm = 0$. The nontriviality of the flat (0,1)-connection arises from the gluing condition on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$ which reads as

$$\psi_+\bar\partial\psi_+^{-1} = \psi_-\bar\partial\psi_-^{-1}\,, \tag{2.48}$$

since $\mathcal{E}$ is a trivial bundle. Upon using the identity $\bar\partial\psi_\pm^{-1} = -\psi_\pm^{-1}(\bar\partial\psi_\pm)\psi_\pm^{-1}$, one obtains

$$\bar\partial(\psi_+^{-1}\psi_-) = 0 \tag{2.49}$$

[10] This theory was introduced in [4].

[11] This subset contains in particular the vacuum solution $\mathcal{A}^{0,1} = 0$, and therefore hCS theory is perturbatively equivalent to $\mathcal{N} = 4$ SDYM theory.

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and thus, we can define a holomorphic vector bundle $\tilde{\mathcal{E}} \to \mathcal{P}^{3|4}$ with the holomorphic structure $\bar{\partial}$ and a transition function

$$\tilde{f}_{+-} := \psi_+^{-1}\psi_- \,. \tag{2.50}$$

One should stress that the bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are diffeomorphic but not biholomorphic. In this description, the above assumed existence of a gauge in which $\mathcal{A}_{\bar{\lambda}_\pm} = 0$ is equivalent to the holomorphic triviality of the bundle $\tilde{\mathcal{E}}$ when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|4}$.

Note that the condition

$$\begin{aligned} \psi_+(x, \eta_i^{\dot\alpha}, \lambda_+) \;&=\; \left(\psi_-^{-1}(\tau(x, \eta_i^{\dot\alpha}, \lambda_-))\right)^\dagger \,, \\ \Leftrightarrow \qquad \tilde{f}_{+-}(x, \eta_i^{\dot\alpha}, \lambda_+) \;&=\; \left(\tilde{f}_{+-}\left(\tau(x, \eta_i^{\dot\alpha}, \lambda_+)\right)\right)^\dagger \end{aligned} \tag{2.51}$$

corresponds in the Dolbeault description to the fact that all the fields in the supermultiplet $(f_{\alpha\beta}, \chi^i_\alpha, \phi^{ij}, \tilde{\chi}_{i\dot\alpha}, G_{\dot\alpha\dot\beta})$ take values in the Lie algebra $\mathrm{u}(n)$ (or $\mathrm{su}(n)$ if $\det \tilde{f}_{+-} = 1$).

Summarizing, there is a one-to-one correspondence between gauge equivalence classes of solutions to the $\mathcal{N} = 4$ SDYM equations on $(\mathbb{R}^4, \delta_{\mu\nu})$ and equivalence classes of holomorphic vector bundles $\tilde{\mathcal{E}}$ over the supertwistor space $\mathcal{P}^{3|4}$, which become holomorphically trivial when restricted to any τ-invariant projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|4}$. Furthermore, because of the equivalence of the data $(\tilde{\mathcal{E}}, \tilde{f}_{+-}, \bar\partial)$ and $(\mathcal{E}, f_{+-} = \mathbb{1}_n, \bar\partial_{\mathcal{A}})$, the moduli space of hCS theory on $(\mathcal{P}^{3|4}, \tau)$ is bijective to the moduli space of $\mathcal{N} = 4$ SDYM theory on $(\mathbb{R}^4, \delta_{\mu\nu})$.

3. Cauchy-Riemann supermanifolds and mini-supertwistors

The supersymmetric Bogomolny monopole equations are obtained from the four-dimensional supersymmetric self-dual Yang-Mills equations by the dimensional reduction $\mathbb{R}^4 \to \mathbb{R}^3$. In this section, we study in detail the meaning of this reduction on the level of the supertwistor space. We find that the supertwistor space $\mathcal{P}^{3|4}$, when interpreted as the real manifold $\mathbb{R}^{4|8} \times S^2$, reduces to the space $\mathbb{R}^{3|8} \times S^2$. As a complex manifold, however, $\mathcal{P}^{3|4}$ reduces to the rank 1|4 holomorphic vector bundle $\mathcal{P}^{2|4} := \mathcal{O}(2) \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^4$ over $\mathbb{C}P^1$. Due to this difference, the twistor correspondence gets more involved (one needs a double fibration). We also show that on the space $\mathbb{R}^{3|8} \times S^2$, one can introduce a so-called Cauchy-Riemann structure, which generalizes the notion of a complex structure. This allows us to use tools familiar from complex geometry.

In later sections, we will then discuss the supertwistor description of the supersymmetric Bogomolny equations and, in particular, the reductions of hCS theory (2.44) and (2.45) to theories on $\mathcal{P}^{2|4}$ and $\mathbb{R}^{3|8} \times S^2$, which lay the ground for future calculations using twistor string techniques.

3.1 The dimensional reduction $\mathbb{R}^{4|8} \times S^2 \to \mathbb{R}^{3|8} \times S^2$

Supersymmetric Bogomolny equations. It is well known that the Bogomolny equations on $\mathbb{R}^3$ describing BPS monopoles [30, 31] can be obtained from the SDYM equations on $\mathbb{R}^4$ by demanding the components A_μ with $\mu = 1, \ldots, 4$ of a gauge potential to be independent of x^4 and by putting $\Phi := A_4$ [32, 15, 33]. Here, Φ is a Lie-algebra valued scalar field in three dimensions (the Higgs field) which enters into the Bogomolny equations. Obviously, one can similarly reduce the $\mathcal{N}=4$ SDYM equations (2.46) on $\mathbb{R}^4$ by imposing the $\frac{\partial}{\partial x^4}$-invariance condition on all the fields $(f_{\alpha\beta}, \chi^i_\alpha, \phi^{ij}, \tilde\chi_{i\dot\alpha}, G_{\dot\alpha\dot\beta})$ in the supermultiplet and obtain supersymmetric Bogomolny equations on $\mathbb{R}^3$. Recall that both $\mathcal{N}=4$ SDYM theory and $\mathcal{N}=4$ SYM theory have an $\mathrm{SU}(4)\cong\mathrm{Spin}(6)$ R-symmetry group. In the case of the full $\mathcal{N}=4$ super Yang-Mills theory, the R-symmetry group and supersymmetry get enlarged to Spin(7) and $\mathcal{N}=8$ supersymmetry by a reduction from four to three dimensions, cf. [1]. However, the situation in the dimensionally reduced $\mathcal{N}=4$ SDYM theory is more involved since there is no parity symmetry interchanging left-handed and right-handed fields, and only the SU(4) subgroup of Spin(7) is manifest as an R-symmetry of the Bogomolny model.

The reduction $\mathbb{R}^{4|8}\to\mathbb{R}^{3|8}$. Recall that on $\mathbb{R}^4\cong\mathbb{C}^2$, we may use the complex coordinates $x^{\alpha\dot\alpha}$ satisfying the reality conditions (2.15) or the real coordinates x^μ defined in (2.16). Translations generated by the vector field $\mathscr{T}_4 := \frac{\partial}{\partial x^4}$ are isometries of $\mathbb{R}^{4|8}$ and by taking the quotient with respect to the action of the abelian group $\mathscr{G} := \{\exp(a\mathscr{T}_4) : x^4\mapsto x^4+a, a\in\mathbb{R}\}$ generated by $\mathscr{T}_4$, we obtain the superspace $\mathbb{R}^{3|8}\cong\mathbb{R}^{4|8}/\mathscr{G}$. Recall that the eight odd complex coordinates $\eta^{\dot\alpha}_i$ satisfy the reality conditions (2.33). The vector field $\mathscr{T}_4$ is trivially lifted to $\mathbb{R}^{4|8}\times S^2$ (see e.g. [11]) and therefore the supertwistor space, considered as the smooth supermanifold $\mathbb{R}^{4|8}\times S^2$, is reduced to $\mathbb{R}^{3|8}\times S^2\cong\mathbb{R}^{4|8}\times S^2/\mathscr{G}$. In other words, smooth $\mathscr{T}_4$-invariant functions on $\mathcal{P}^{3|4}\cong\mathbb{R}^{4|8}\times S^2$ can be considered as "free" smooth functions on the supermanifold $\mathbb{R}^{3|8}\times S^2$.

Bosonic coordinates on $\mathbb{R}^{3|8}$. Recall that the rotation group SO(4) of $(\mathbb{R}^4,\delta_{\mu\nu})$ is locally isomorphic to $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\cong\mathrm{Spin}(4)$. Upon dimensional reduction to three dimensions, the rotation group SO(3) of $(\mathbb{R}^3,\delta_{rs})$ with $r,s=1,2,3$ is locally $\mathrm{SU}(2)\cong\mathrm{Spin}(3)$, which is the diagonal group $\mathrm{diag}(\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R)$. Therefore, the distinction between undotted, i.e. $\mathrm{SU}(2)_L$, and dotted, i.e. $\mathrm{SU}(2)_R$, indices disappears. This implies, that one can relabel the bosonic coordinates $x^{\alpha\dot\beta}$ from (2.15), (2.16) by $x^{\dot\alpha\dot\beta}$ and split them as

$$x^{\dot\alpha\dot\beta} = x^{(\dot\alpha\dot\beta)} + x^{[\dot\alpha\dot\beta]} \ := \ \frac{1}{2}(x^{\dot\alpha\dot\beta}+x^{\dot\beta\dot\alpha}) + \frac{1}{2}(x^{\dot\alpha\dot\beta}-x^{\dot\beta\dot\alpha}), \tag{3.1}$$

into symmetric

$$y^{\dot\alpha\dot\beta} := -\mathrm{i}x^{(\dot\alpha\dot\beta)} \qquad \text{with} \qquad y^{\dot1\dot1} \ = \ -\bar y^{\dot2\dot2} = (x^1+\mathrm{i}x^2) =: y\,, \qquad y^{\dot1\dot2} = \bar y^{\dot1\dot2} = -x^3 \tag{3.2}$$

and antisymmetric

$$x^{[\dot\alpha\dot\beta]} = \varepsilon^{\dot\alpha\dot\beta}x^4 \tag{3.3}$$

parts. More abstractly, this splitting corresponds to the decomposition $\mathbf{4}=\mathbf{3}\oplus\mathbf{1}$ of the irreducible real vector representation $\mathbf{4}$ of the group $\mathrm{Spin}(4)\cong\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ into two irreducible real representations $\mathbf{3}$ and $\mathbf{1}$ of the group $\mathrm{Spin}(3)\cong\mathrm{SU}(2)=\mathrm{diag}(\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R)$. For future use, we also introduce the operators

$$\partial_{(\dot\alpha\dot\beta)} \ := \ \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial x^{\dot\alpha\dot\beta}}+\frac{\partial}{\partial x^{\dot\beta\dot\alpha}}\right), \tag{3.4a}$$

which read explicitly as

$$\partial_{(\mathrm{i}\mathrm{i})} = \frac{\partial}{\partial y^{\mathrm{i}\mathrm{i}}}\,, \qquad \partial_{(\mathrm{i}\dot{2})} = \frac{1}{2}\frac{\partial}{\partial y^{\mathrm{i}\dot{2}}} \qquad \text{and} \qquad \partial_{(\dot{2}\dot{2})} = \frac{\partial}{\partial y^{\dot{2}\dot{2}}}\,. \tag{3.4b}$$

Altogether, we thus have

$$\frac{\partial}{\partial x^{\dot\alpha\dot\beta}} = -\mathrm{i}\partial_{(\dot\alpha\dot\beta)} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\frac{\partial}{\partial x^4}\,. \tag{3.5}$$

3.2 The holomorphic reduction $\mathcal{P}^{3|4} \to \mathcal{P}^{2|4}$

The complex abelian group action. The vector field $\mathscr{T}_4 = \frac{\partial}{\partial x^4}$ yields a free twistor space action of the abelian group $\mathscr{G} \cong \mathbb{R}$ which is the real part of the holomorphic action of the complex group $\mathscr{G}_{\mathbb{C}} \cong \mathbb{C}$. In other words, we have

$$\begin{aligned} \mathscr{T}_4 = \frac{\partial}{\partial x^4} &= \frac{\partial z_+^a}{\partial x^4}\frac{\partial}{\partial z_+^a} + \frac{\partial \bar z_+^a}{\partial x^4}\frac{\partial}{\partial \bar z_+^a} \\ &= \left(-\frac{\partial}{\partial z_+^2} + z_+^3\frac{\partial}{\partial z_+^1}\right) + \left(-\frac{\partial}{\partial \bar z_+^2} + \bar z_+^3\frac{\partial}{\partial \bar z_+^1}\right) =: \mathscr{T}_+' + \bar{\mathscr{T}}_+' \end{aligned} \tag{3.6}$$

in the coordinates (z_+^a, η_i^+) on $\hat{\mathcal{U}}_+$, where

$$\mathscr{T}_+' := \mathscr{T}'|_{\hat{\mathcal{U}}_+} = -\frac{\partial}{\partial z_+^2} + z_+^3\frac{\partial}{\partial z_+^1} \tag{3.7}$$

is a holomorphic vector field on $\hat{\mathcal{U}}_+$. Similarly, we obtain

$$\mathscr{T}_4 = \mathscr{T}_-' + \bar{\mathscr{T}}_-' \qquad \text{with} \qquad \mathscr{T}_-' \;:=\; \mathscr{T}'|_{\hat{\mathcal{U}}_-} = -z_-^3\frac{\partial}{\partial z_-^2} + \frac{\partial}{\partial z_-^1} \tag{3.8}$$

on $\hat{\mathcal{U}}_-$ and $\mathscr{T}_+' = \mathscr{T}_-'$ on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. For holomorphic functions f on $\mathcal{P}^{3|4}$ we have

$$\mathscr{T}_4 f(z_\pm^a, \eta_i^\pm) = \mathscr{T}' f(z_\pm^a, \eta_i^\pm) \tag{3.9}$$

and therefore $\mathscr{T}'$-invariant holomorphic functions on $\mathcal{P}^{3|4}$ can be considered as "free" holomorphic functions on a reduced space $\mathcal{P}^{2|4} \cong \mathcal{P}^{3|4}/\mathscr{G}_{\mathbb{C}}$ obtained as the quotient space of $\mathcal{P}^{3|4}$ by the action of the complex abelian group $\mathscr{G}_{\mathbb{C}}$ generated by $\mathscr{T}'$.

Reduction diagram. In the purely bosonic case, the space $\mathcal{P}^2 \cong \mathcal{P}^3/\mathscr{G}_{\mathbb{C}}$ was called mini-twistor space [15] and we shall refer to $\mathcal{P}^{2|4}$ as the *mini-supertwistor space*. To sum up, the reduction of the supertwistor correspondence induced by the $\mathscr{T}_4$-action is described by the following diagram:

$$\begin{tikzcd} \mathcal{P}^{3|4} \arrow[ddd] & \cong & \mathbb{R}^{4|8}\times S^2 \arrow[r] \arrow[d] & \mathbb{R}^{4|8} \arrow[ddd] \\ & & \mathbb{R}^{3|8}\times S^2 \arrow[ddll, "\pi_2"'] \arrow[ddr, "\pi_1"] & \\ & & & \\ \mathcal{P}^{2|4} & & & \mathbb{R}^{3|8} \end{tikzcd} \tag{3.10}$$

Here, $\downarrow$ symbolizes projections generated by the action of the groups $\mathscr{G}$ or $\mathscr{G}_{\mathbb{C}}$ and π_1 is the canonical projection. The projection π_2 will be described momentarily.

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3.3 Geometry of the mini-supertwistor space

Local coordinates. It is not difficult to see that the functions

$$\begin{aligned} w_+^1 &:= -\mathrm{i}(z_+^1 + z_+^3 z_+^2)\,, \qquad w_+^2 := z_+^3 \qquad \text{and} \qquad \eta_i^+ \qquad \text{on} \qquad \hat{\mathcal{U}}_+\,, \\ w_-^1 &:= -\mathrm{i}(z_-^2 + z_-^3 z_-^1)\,, \qquad w_-^2 := z_-^3 \qquad \text{and} \qquad \eta_i^- \qquad \text{on} \qquad \hat{\mathcal{U}}_- \end{aligned} \tag{3.11}$$

are constant along the $\mathscr{G}_{\mathbb{C}}$-orbits in $\mathcal{P}^{3|4}$ and thus descend to the patches $\hat{\mathcal{V}}_\pm := \hat{\mathcal{U}}_\pm \cap \mathcal{P}^{2|4}$ covering the (orbit) space $\mathcal{P}^{2|4} \cong \mathcal{P}^{3|4}/\mathscr{G}_{\mathbb{C}}$. On the overlap $\hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}_-$, we have

$$w_+^1 = \frac{1}{(w_-^2)^2} w_-^1\,, \qquad w_+^2 = \frac{1}{w_-^2} \qquad \text{and} \qquad \eta_i^+ = \frac{1}{w_-^2}\eta_i^- \tag{3.12}$$

which coincides with the transformation of canonical coordinates on the total space

$$\mathcal{O}(2) \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^4 = \mathcal{P}^{2|4} \tag{3.13}$$

of the holomorphic vector bundle

$$\mathcal{P}^{2|4} \;\to\; \mathbb{C}P^1\,. \tag{3.14}$$

This space is a Calabi-Yau supermanifold [1] with a holomorphic volume form

$$\Omega|_{\hat{\mathcal{V}}_\pm} := \pm \mathrm{d}w_\pm^1 \wedge \mathrm{d}w_\pm^2 \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm\,. \tag{3.15}$$

The body of this Calabi-Yau supermanifold is the mini-twistor space [15]

$$\mathcal{P}^2 \;\cong\; \mathcal{O}(2) \;\cong\; T^{1,0}\mathbb{C}P^1\,, \tag{3.16}$$

where $T^{1,0}\mathbb{C}P^1$ denotes the holomorphic tangent bundle of $\mathbb{C}P^1$. Note that the space $\mathcal{P}^{2|4}$ can be considered as an open subset of the weighted projective space $W\mathbb{C}P^{2|4}(2,1,1|1,1,1,1)$.

Incidence relations. The real structure τ on $\mathcal{P}^{3|4}$ induces a real structure on $\mathcal{P}^{2|4}$ acting on local coordinates by the formula

$$\tau(w_\pm^1, w_\pm^2, \eta_i^\pm) = \left(-\frac{\bar{w}_\pm^1}{(\bar{w}_\pm^2)^2}, -\frac{1}{\bar{w}_\pm^2}, \pm\frac{1}{\bar{w}_\pm^2} T_i{}^j \bar{\eta}_j^\pm \right)\,, \tag{3.17}$$

where the matrix $T = (T_i{}^j)$ has already been defined in (2.31). From (3.17), one sees that τ has no fixed points in $\mathcal{P}^{2|4}$ but leaves invariant projective lines $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{2|4}$ defined by the equations

$$\begin{aligned} w_+^1 &= y - 2\lambda_+ x^3 - \lambda_+^2 \bar{y}\,, \qquad \eta_i^+ = \eta_i^{\dot 1} + \lambda_+ \eta_i^{\dot 2} \qquad \text{with} \;\; \lambda_+ = w_+^2 \;\in\; U_+\,, \\ w_-^1 &= \lambda_-^2 y - 2\lambda_- x^3 - \bar{y}\,, \qquad \eta_i^- = \lambda_- \eta_i^{\dot 1} + \eta_i^{\dot 2} \qquad \text{with} \;\; \lambda_- = w_-^2 \;\in\; U_- \end{aligned} \tag{3.18}$$

for fixed $(x,\eta) \in \mathbb{R}^{3|8}$. Here, $y = x^1 + \mathrm{i}x^2$, $\bar{y} = x^1 - \mathrm{i}x^2$ and x^3 are coordinates on $\mathbb{R}^3$.

By using the coordinates (3.2), we can rewrite (3.18) as

$$w_\pm^1 = \lambda_{\dot\alpha}^\pm \lambda_{\dot\beta}^\pm y^{\dot\alpha\dot\beta}\,, \qquad w_\pm^2 = \lambda_\pm \qquad \text{and} \qquad \eta_i^\pm = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm\,, \tag{3.19}$$

where the explicit form of $\lambda^{\pm}_{\dot\alpha}$ has been given in (2.12). In fact, the equations (3.19) are the incidence relations which lead to the double fibration

$$\begin{tikzcd} & \mathcal{F}^{5|8} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}^{2|4} & & \mathbb{R}^{3|8} \end{tikzcd} \tag{3.20}$$

where $\mathcal{F}^{5|8} \cong \mathbb{R}^{3|8} \times S^2$, π_1 is again the canonical projection onto $\mathbb{R}^{3|8}$ and the projection π_2 is defined by the formula

$$\pi_2(x^r, \lambda_\pm, \eta_i^{\dot\alpha}) = \pi_2(y^{\dot\alpha\dot\beta}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha}) = (w^1_\pm, w^2_\pm, \eta_i^\pm)\,, \tag{3.21}$$

where $r = 1, 2, 3$, and $w^{1,2}_\pm$ and $\eta^\pm_i$ are given in (3.19). The diagram (3.20), which is a part of (3.10), describes the one-to-one correspondences

$$\begin{aligned} &\{\,\tau\text{-invariant projective lines } \mathbb{C}P^1_{x,\eta} \text{ in } \mathcal{P}^{2|4}\} \;\longleftrightarrow\; \{\,\text{points } (x,\eta) \text{ in } \mathbb{R}^{3|8}\}\,, \\ &\{\,\text{points } p \text{ in } \mathcal{P}^{2|4}\} \;\longleftrightarrow\; \{\,\text{oriented } (1|0)\text{-dimensional lines } \ell_p \text{ in } \mathbb{R}^{3|8}\}\,. \end{aligned} \tag{3.22}$$

3.4 Cauchy-Riemann supertwistors

Cauchy-Riemann structure. Consider the double fibration (3.20). For a moment, let us restrict ourselves to the purely bosonic setup. The body of the space $\mathbb{R}^{3|8} \times S^2$ is the real five-dimensional manifold $\mathbb{R}^3 \times S^2$. As an odd-dimensional space, this is obviously not a complex manifold, but it can be understood as a so-called *Cauchy-Riemann* (CR) *manifold*, i.e. as a *partially complex manifold*. Recall that a *CR structure* on a smooth manifold X of real dimension d is a complex subbundle $\bar{\mathscr{D}}$ of rank m of the complexified tangent bundle $T_{\mathbb{C}}X$ such that $\mathscr{D} \cap \bar{\mathscr{D}} = \{0\}$ and $\bar{\mathscr{D}}$ is involutive (integrable), i.e., the space of smooth sections of $\bar{\mathscr{D}}$ is closed under the Lie bracket. Obviously, the distribution[12] $\mathscr{D}$ is integrable if $\bar{\mathscr{D}}$ is integrable. The pair $(X, \bar{\mathscr{D}})$ is called a CR manifold of dimension $d = \dim_{\mathbb{R}} X$, of rank $m = \dim_{\mathbb{C}} \bar{\mathscr{D}}$ and of codimension $d - 2m$. In particular, a CR structure on X in the special case $d = 2m$ is a complex structure on X. Thus, the notion of CR manifolds generalizes that of complex manifolds.

On the manifold $\mathbb{R}^3 \times S^2$, one can introduce several CR structures. One of them, which we denote by $\bar{\mathscr{D}}_0$, is generated by the vector fields $\{\partial_{\bar y}, \partial_{\bar\lambda_\pm}\}$ and corresponds to the identification $\mathcal{F}^5_0 := (\mathbb{R}^3 \times S^2, \bar{\mathscr{D}}_0) \cong \mathbb{R} \times \mathbb{C} \times \mathbb{C}P^1$. Another one, denoted by $\bar{\mathscr{D}}$, is spanned by the basis sections $\{\partial_{\bar w^1_\pm}, \partial_{\bar w^2_\pm}\}$ of the bundle $T_{\mathbb{C}}(\mathbb{R}^3 \times S^2)$. Moreover, $\mathscr{D} \cap \bar{\mathscr{D}} = \{0\}$ and the distribution $\bar{\mathscr{D}}$ is integrable since $[\partial_{\bar w^1_\pm}, \partial_{\bar w^2_\pm}] = 0$. Therefore, the pair $(\mathbb{R}^3 \times S^2, \bar{\mathscr{D}}) =: \mathcal{F}^5$ is also a CR manifold. It is obvious that there is a diffeomorphism between the manifolds $\mathcal{F}^5$ and $\mathcal{F}^5_0$, but this is not a CR diffeomorphism since it does not respect the chosen CR structures. Note that a CR five-manifold generalizing the above manifold $\mathcal{F}^5$ can be constructed as a sphere bundle over an arbitrary three-manifold with conformal metric [21]. Following [21], we shall call $\mathcal{F}^5$ the *CR twistor space*.

So far, we have restricted our attention to the purely bosonic setup. However, the above definitions carry naturally over to the case of supermanifolds (see e.g. [34]). Namely, by considering the integrable distribution $\hat{\bar{\mathscr{D}}}$ generated by the vector fields $\partial_{\bar w^1_\pm}$, $\partial_{\bar w^2_\pm}$ and

[12] We use the same letter for the bundle $\mathscr{D}$ and a distribution generated by its sections.

$\partial_{\bar{\eta}_i^\pm}$, we obtain the CR supertwistor space

$$\mathcal{F}^{5|8} := (\mathbb{R}^{3|8} \times S^2, \hat{\bar{\mathscr{D}}}) \tag{3.23}$$

with

$$\hat{\bar{\mathscr{D}}} = \text{span}\left\{ \frac{\partial}{\partial \bar{w}_\pm^1}, \frac{\partial}{\partial \bar{w}_\pm^2}, \frac{\partial}{\partial \bar{\eta}_i^\pm} \right\}. \tag{3.24}$$

Similarly, we obtain the CR supermanifold $\mathcal{F}_0^{5|8} := (\mathbb{R}^{3|8} \times S^2, \hat{\bar{\mathscr{D}}}_0)$ for the distribution $\hat{\bar{\mathscr{D}}}_0 = \text{span}\{\frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{\lambda}_\pm}, \frac{\partial}{\partial \bar{\eta}_i^1}\}$. In both cases, the CR structures have rank $2|4$.

Real and complex coordinates on $\mathcal{F}^{5|8}$. Up to now, we have used the coordinates $(y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot{\alpha}})$ or $(y^{\dot{\alpha}\dot{\beta}}, \lambda_{\dot{\alpha}}^\pm, \hat{\lambda}_{\dot{\alpha}}^\pm, \eta_i^{\dot{\alpha}})$ on the two patches $\tilde{\mathcal{V}}_\pm$ covering the superspace $\mathbb{R}^{3|8} \times S^2$. More appropriate for the distribution (3.24) are, however, the coordinates (3.18) together with

$$\begin{aligned} w_+^3 &:= \frac{1}{1+\lambda_+\bar{\lambda}_+}\left[\bar{\lambda}_+ y + (1-\lambda_+\bar{\lambda}_+)x^3 + \lambda_+\bar{y}\right] \qquad \text{on } \tilde{\mathcal{V}}_+\,, \\ w_-^3 &:= \frac{1}{1+\lambda_-\bar{\lambda}_-}\left[\lambda_- y + (\lambda_-\bar{\lambda}_- - 1)x^3 + \bar{\lambda}_-\bar{y}\right] \qquad \text{on } \tilde{\mathcal{V}}_-\,. \end{aligned} \tag{3.25}$$

In terms of the coordinates (3.2) and $\lambda_{\dot{\alpha}}^\pm$ from (2.12), we can rewrite (3.18) and (3.25) concisely as

$$w_\pm^1 = \lambda_{\dot{\alpha}}^\pm \lambda_{\dot{\beta}}^\pm y^{\dot{\alpha}\dot{\beta}}\,, \qquad w_\pm^2 = \lambda_\pm\,, \qquad w_\pm^3 = -\gamma_\pm \lambda_{\dot{\alpha}}^\pm \hat{\lambda}_{\dot{\beta}}^\pm y^{\dot{\alpha}\dot{\beta}} \qquad \text{and} \qquad \eta_i^\pm = \eta_i^{\dot{\alpha}} \lambda_{\dot{\alpha}}^\pm\,, \tag{3.26}$$

where the factors $\gamma_\pm$ have been given in (2.22). Note that the coordinates $w_\pm^3$ are real and all the other coordinates in (3.26) are complex. The relations between the coordinates on $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$ follow directly from their definitions (3.26).

Projection onto $\mathcal{P}^{2|4}$. The coordinates $w_\pm^{1,2}$ and $\eta_i^\pm$ have already appeared in (3.19) since $\mathcal{P}^{2|4}$ is a complex subsupermanifold of $\mathcal{F}^{5|8}$. Recall that the formulæ (3.21) together with (3.26) define a projection

$$\pi_2 \;:\; \mathcal{F}^{5|8} \;\to\; \mathcal{P}^{2|4} \tag{3.27}$$

onto the mini-supertwistor space $\mathcal{P}^{2|4}$. The fibres over points $p \in \mathcal{P}^{2|4}$ of this projection are real one-dimensional spaces $\ell_p \cong \mathbb{R}$ parametrized by the coordinates $w_\pm^3$. Note that the pull-back to $\mathcal{F}^{5|8}$ of the real structure τ on $\mathcal{P}^{2|4}$ given in (3.17) reverses the orientation of each line ℓ_p, since $\tau(w_\pm^3) = -w_\pm^3$.

In order to clarify the geometry of the fibration (3.27), we note that the body $\mathcal{F}^5 \cong \mathbb{R}^3 \times S^2$ of the supermanifold $\mathcal{F}^{5|8}$ can be considered as the sphere bundle

$$S(T\mathbb{R}^3) = \{(x,u) \in T\mathbb{R}^3 \,|\, \delta_{rs} u^r u^s = 1\} \;\cong\; \mathbb{R}^3 \times S^2 \tag{3.28}$$

whose fibres at points $x \in \mathbb{R}^3$ are spheres of unit vectors in $T_x\mathbb{R}^3$ [15]. Since this bundle is trivial, its projection onto $\mathbb{R}^3$ is obviously $\pi_1(x,u) = x$. Moreover, the complex two-dimensional mini-twistor space $\mathcal{P}^2$ can be described as the space of all oriented lines in $\mathbb{R}^3$. That is, any such line ℓ is defined by a unit vector u^r in the direction of ℓ and a shortest vector v^r from the origin in $\mathbb{R}^3$ to ℓ, and one can show [15] that

$$\mathcal{P}^2 = \{(v,u) \in T\mathbb{R}^3 \,|\, \delta_{rs}u^r v^s = 0\,,\ \delta_{rs}u^r u^s = 1\} \ \cong\ T^{1,0}\mathbb{C}P^1 \ \cong\ \mathcal{O}(2)\,. \tag{3.29}$$

The fibres of the projection $\pi_2 : \mathbb{R}^3 \times S^2 \to \mathcal{P}^2$ are the orbits of the action of the group $\mathscr{G}' \cong \mathbb{R}$ on $\mathbb{R}^3 \times S^2$ given by the formula $(v^r, u^s) \mapsto (v^r + tu^r, u^s)$ for $t \in \mathbb{R}$ and

$$\mathcal{P}^2 \ \cong\ \mathbb{R}^3 \times S^2/\mathscr{G}'\,. \tag{3.30}$$

Recall that $\mathcal{F}^5 \cong \mathbb{R}^3 \times S^2$ is a (real) hypersurface in the twistor space $\mathcal{P}^3$. On the other hand, $\mathcal{P}^2$ is a complex two-dimensional submanifold of $\mathcal{F}^5$ and therefore

$$\mathcal{P}^2 \ \subset\ \mathcal{F}^5 \ \subset\ \mathcal{P}^3\,. \tag{3.31}$$

Similarly, we have in the supertwistor case

$$\mathcal{P}^{2|4} \ \subset\ \mathcal{F}^{5|8} \ \subset\ \mathcal{P}^{3|4}\,. \tag{3.32}$$

Vector fields on $\mathcal{F}^{5|8}$. The formulæ (3.18) and (3.25) define a coordinate transformation $(y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot\alpha}) \mapsto (w_\pm^a, \eta_i^\pm)$ on $\mathcal{F}^{5|8}$. From corresponding inverse formulæ defining the transformation $(w_\pm^a, \eta_i^\pm) \mapsto (y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot\alpha})$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial w_+^1} &= \gamma_+^2 \left(\frac{\partial}{\partial y} - \bar{\lambda}_+ \frac{\partial}{\partial x^3} - \bar{\lambda}_+^2 \frac{\partial}{\partial \bar{y}}\right) \ =:\ \gamma_+^2 W_1^+\,,\\
\frac{\partial}{\partial w_+^2} &= W_2^+ + 2\gamma_+^2 (x^3 + \lambda_+ \bar{y}) W_1^+ - \gamma_+^2 (\bar{y} - 2\bar{\lambda}_+ x^3 - \bar{\lambda}_+^2 y) W_3^+ - \gamma_+ \bar{\eta}_i^{\dot 1} V_+^i\,,\\
\frac{\partial}{\partial w_+^3} &= 2\gamma_+ \left(\lambda_+ \frac{\partial}{\partial y} + \bar{\lambda}_+ \frac{\partial}{\partial \bar{y}} + \frac{1}{2}(1 - \lambda_+ \bar{\lambda}_+) \frac{\partial}{\partial x^3}\right) \ =:\ W_3^+\,,
\end{aligned} \tag{3.33a}$$

as well as

$$\begin{aligned}
\frac{\partial}{\partial w_-^1} &= \gamma_-^2 \left(\bar{\lambda}_-^2 \frac{\partial}{\partial y} - \bar{\lambda}_- \frac{\partial}{\partial x^3} - \frac{\partial}{\partial \bar{y}}\right) =: \gamma_-^2 W_1^-\,,\\
\frac{\partial}{\partial w_-^2} &= W_2^- + 2\gamma_-^2 (x^3 - \lambda_- y) W_1^- + \gamma_-^2 (\bar{\lambda}_-^2 \bar{y} - 2\bar{\lambda}_- x^3 - y) W_3^- + \gamma_- \bar{\eta}_i^{\dot 2} V_-^i\,,\\
\frac{\partial}{\partial w_-^3} &= 2\gamma_- \left(\bar{\lambda}_- \frac{\partial}{\partial y} + \lambda_- \frac{\partial}{\partial \bar{y}} + \frac{1}{2}(\lambda_- \bar{\lambda}_- - 1) \frac{\partial}{\partial x^3}\right) \ =:\ W_3^-\,,
\end{aligned} \tag{3.33b}$$

where $W_2^\pm := \frac{\partial}{\partial \lambda_\pm}$. Thus, when working in the coordinates $(y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot\alpha})$ on $\tilde{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$, we will use the bosonic vector fields $W_a^\pm$ with $a = 1, 2, 3$ and the fermionic vector fields $V_\pm^i$ together with their complex conjugates[13] $\bar{W}_{1,2}^\pm$ and $\bar{V}_\pm^i$, respectively.

In the coordinates $(y^{\dot\alpha\dot\beta}, \lambda_{\dot\alpha}^\pm, \hat{\lambda}_{\dot\alpha}^\pm, \eta_i^{\dot\alpha})$, the vector fields $W_a^\pm$, $V_\pm^i$ and their complex conjugates read as

$$\begin{aligned}
W_1^\pm = \hat{\lambda}_\pm^{\dot\alpha} \hat{\lambda}_\pm^{\dot\beta} \partial_{(\dot\alpha\dot\beta)}\,, \qquad W_2^\pm = \partial_{\lambda_\pm}\,, \qquad W_3^\pm = 2\gamma_\pm \hat{\lambda}_\pm^{\dot\alpha} \lambda_\pm^{\dot\beta} \partial_{(\dot\alpha\dot\beta)}\,,\\
V_\pm^i = -\hat{\lambda}_\pm^{\dot\alpha} T_j{}^i \frac{\partial}{\partial \eta_j^{\dot\alpha}}\,,
\end{aligned} \tag{3.34a}$$

[13]Note that the vector field $W_3^\pm$ is real.

as well as

$$\bar{W}_1^\pm = -\lambda_\pm^{\dot\alpha}\lambda_\pm^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\,, \qquad \bar{W}_2^\pm = \partial_{\bar\lambda_\pm}\,, \qquad \bar{W}_3^\pm = W_3^\pm = 2\gamma_\pm\hat\lambda_\pm^{\dot\alpha}\lambda_\pm^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\,,$$
$$\bar{V}_\pm^i = \lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_i^{\dot\alpha}}\,, \tag{3.34b}$$

where

$$(\hat\lambda_+^{\dot\alpha}) = \begin{pmatrix} 1 \\ \bar\lambda_+ \end{pmatrix} \qquad \text{and} \qquad (\hat\lambda_-^{\dot\alpha}) = \begin{pmatrix} \bar\lambda_- \\ 1 \end{pmatrix}. \tag{3.35}$$

Recall that the vector fields $\bar{W}_1^\pm$, $\bar{W}_2^\pm$ and $\bar{V}_\pm^i$ generate a distribution $\hat{\bar{\mathscr{D}}}$ (CR structure) on $\mathbb{R}^{3|8}\times S^2$, which is obviously integrable as these vector fields commute with each other.

Forms on $\mathcal{F}^{5|8}$. It is not difficult to see that forms dual to the vector fields (3.34a) and (3.34b) are

$$\Theta_\pm^1 := \gamma_\pm^2\lambda_{\dot\alpha}^\pm\lambda_{\dot\beta}^\pm \mathrm{d}y^{\dot\alpha\dot\beta}\,, \qquad \Theta_\pm^2 := \mathrm{d}\lambda_\pm\,, \qquad \Theta_\pm^3 := -\gamma_\pm\lambda_{\dot\alpha}^\pm\hat\lambda_{\dot\beta}^\pm \mathrm{d}y^{\dot\alpha\dot\beta}\,,$$
$$E_i^\pm := \gamma_\pm\lambda_{\dot\alpha}^\pm T_i{}^j\mathrm{d}\eta_j^{\dot\alpha} \tag{3.36a}$$

and

$$\bar\Theta_\pm^1 = -\gamma_\pm^2\hat\lambda_{\dot\alpha}^\pm\hat\lambda_{\dot\beta}^\pm \mathrm{d}y^{\dot\alpha\dot\beta}\,, \qquad \bar\Theta_\pm^2 = \mathrm{d}\bar\lambda_\pm\,, \qquad \bar\Theta_\pm^3 = \Theta_\pm^3\,,$$
$$\bar{E}_i^\pm = -\gamma_\pm\hat\lambda_{\dot\alpha}^\pm\mathrm{d}\eta_i^{\dot\alpha}\,, \tag{3.36b}$$

where $T_i{}^j$ has been given in (2.31). The exterior derivative on $\mathcal{F}^{5|8}$ reads as

$$\begin{aligned} \mathrm{d}|_{\tilde{\mathcal{V}}_\pm} &= \mathrm{d}w_\pm^1\frac{\partial}{\partial w_\pm^1} + \mathrm{d}w_\pm^2\frac{\partial}{\partial w_\pm^2} + \mathrm{d}\bar{w}_\pm^1\frac{\partial}{\partial \bar{w}_\pm^1} + \mathrm{d}\bar{w}_\pm^2\frac{\partial}{\partial \bar{w}_\pm^2} + \mathrm{d}w_\pm^3\frac{\partial}{\partial w_\pm^3} + \mathrm{d}\eta_i^\pm\frac{\partial}{\partial \eta_i^\pm} + \mathrm{d}\bar\eta_i^\pm\frac{\partial}{\partial \bar\eta_i^\pm} \\ &= \Theta_\pm^1 W_1^\pm + \Theta_\pm^2 W_2^\pm + \bar\Theta_\pm^1\bar{W}_1^\pm + \bar\Theta_\pm^2\bar{W}_2^\pm + \Theta_\pm^3 W_3^\pm + E_i^\pm V_\pm^i + \bar{E}_i^\pm\bar{V}_\pm^i\,. \end{aligned} \tag{3.37}$$

Note again that $\Theta_\pm^3$ and $W_3^\pm$ are both real.[14]

4. Partially holomorphic Chern-Simons theory

We have discussed how the mini-supertwistor and CR supertwistor spaces arise via dimensional reductions from the supertwistor space of four-dimensional spacetime. Subject of this section is the discussion of a generalization of Chern-Simons theory and its relatives to this setup. We call the theory we are about to introduce *partially holomorphic Chern-Simons theory* or phCS theory for short. Roughly speaking, this theory is a mixture of Chern-Simons and holomorphic Chern-Simons theory on the CR supertwistor space $\mathcal{F}^{5|8}$ which has one real and two complex bosonic dimensions. This theory is a reduction of hCS theory on $\mathcal{P}^{3|4}$. As we will show below, there is a one-to-one correspondence between the moduli space of solutions to the equations of motion of phCS theory on $\mathcal{F}^{5|8}$ and the moduli space of solutions to $\mathcal{N}=4$ supersymmetric Bogomolny equations on $\mathbb{R}^3$, quite similar to the correspondence which exists between hCS theory on the supertwistor space $\mathcal{P}^{3|4}$ and $\mathcal{N}=4$ SDYM theory in four dimensions [2, 3].

[14] To homogenize the notation later on, we shall also use $\bar{W}_3^\pm$ and $\partial_{\bar{w}_3^\pm}$ instead of $W_3^\pm$ and $\partial_{w_3^\pm}$, respectively.

4.1 Partially flat connections

In this subsection, we restrict ourselves to the purely bosonic case since the extension of all definitions to supermanifolds is straightforward.

Integrable distribution $\mathcal{T}$. Let X be a smooth manifold of real dimension d and $T_{\mathbb{C}}X$ the complexified tangent bundle of X. A subbundle $\mathcal{T}\subset T_{\mathbb{C}}X$ is integrable if $i)$ $\mathcal{T}\cap\bar{\mathcal{T}}$ has constant rank k and $ii)$ $\mathcal{T}$ and $\mathcal{T}\cap\bar{\mathcal{T}}$ are closed under the Lie bracket.[15] If $\mathcal{T}$ is such an integrable subbundle of $T_{\mathbb{C}}X$ then X can be covered by open sets and on each open set U, there are coordinates $u^1,\dots,u^l,v^1,\dots,v^k,x^1,\dots,x^m$, $y^1,\dots,y^m$ such that $\mathcal{T}$ is locally spanned by the vector fields

$$\frac{\partial}{\partial v^1},\dots,\frac{\partial}{\partial v^k},\frac{\partial}{\partial \bar{w}^1},\dots,\frac{\partial}{\partial \bar{w}^m}\,, \tag{4.1}$$

where $\bar{w}^1=x^1-\mathrm{i}y^1,\dots,\bar{w}^m=x^m-\mathrm{i}y^m$ [35]. Note that a CR structure is the special case of an integrable subbundle $\mathcal{T}$ with $k=0$.

For any smooth function f on X, let $\mathrm{d}_{\mathcal{T}}f$ denote the restriction of $\mathrm{d}f$ to $\mathcal{T}$, i.e., $\mathrm{d}_{\mathcal{T}}$ is the composition

$$C^\infty(X)\ \xrightarrow{\ \mathrm{d}\ }\ \Omega^1(X)\ \longrightarrow\ \Gamma(X,\mathcal{T}^*)\,, \tag{4.2}$$

where $\Omega^1(X):=\Gamma(X,T^*X)$ and $\mathcal{T}^*$ denotes the sheaf of (smooth) one-forms dual to $\mathcal{T}$ [22]. The operator $\mathrm{d}_{\mathcal{T}}$ can be extended to act on relative q-forms from the space $\Omega^q_{\mathcal{T}}(X):=\Gamma(X,\Lambda^q\mathcal{T}^*)$,

$$\mathrm{d}_{\mathcal{T}}\ :\ \Omega^q_{\mathcal{T}}(X)\ \to\ \Omega^{q+1}_{\mathcal{T}}(X)\,,\qquad \text{for}\qquad q\ \geq\ 0\,. \tag{4.3}$$

Connection on $\mathcal{T}$. Let $\mathcal{E}$ be a smooth complex vector bundle over X. A covariant differential (or connection) on $\mathcal{E}$ along the distribution $\mathcal{T}$ – a $\mathcal{T}$-connection [22] – is a $\mathbb{C}$-linear mapping

$$D_{\mathcal{T}}\ :\ \Gamma(X,\mathcal{E})\ \to\ \Gamma(X,\mathcal{T}^*\otimes\mathcal{E}) \tag{4.4}$$

satisfying the Leibniz formula

$$D_{\mathcal{T}}(f\sigma)=fD_{\mathcal{T}}\sigma+\mathrm{d}_{\mathcal{T}}f\otimes\sigma\,, \tag{4.5}$$

where $\sigma\in\Gamma(X,\mathcal{E})$ is a local section of $\mathcal{E}$ and f is a local smooth function. This $\mathcal{T}$-connection extends to a map

$$D_{\mathcal{T}}\ :\ \Omega^q_{\mathcal{T}}(X,\mathcal{E})\ \to\ \Omega^{q+1}_{\mathcal{T}}(X,\mathcal{E})\,, \tag{4.6}$$

where $\Omega^q_{\mathcal{T}}(X,\mathcal{E}):=\Gamma(X,\Lambda^q\mathcal{T}^*\otimes\mathcal{E})$. Locally, $D_{\mathcal{T}}$ has the form

$$D_{\mathcal{T}}=\mathrm{d}_{\mathcal{T}}+\mathcal{A}_{\mathcal{T}}\,, \tag{4.7}$$

where the standard $\mathrm{End}\,\mathcal{E}$-valued $\mathcal{T}$-connection one-form $\mathcal{A}_{\mathcal{T}}$ has components only along the distribution $\mathcal{T}$. As usual, $D^2_{\mathcal{T}}$ naturally induces

$$\mathcal{F}_{\mathcal{T}}\in\Gamma(X,\Lambda^2\mathcal{T}^*\otimes\mathrm{End}\,\mathcal{E}) \tag{4.8}$$

[15] Again, we use the same letter for the bundle $\mathcal{T}$ and a distribution generated by its sections.

which is the curvature of $\mathcal{A}_\mathcal{T}$. We say that $D_\mathcal{T}$ (or $\mathcal{A}_\mathcal{T}$) is flat, if $\mathcal{F}_\mathcal{T} = 0$. For a flat $D_\mathcal{T}$, the pair $(\mathcal{E}, D_\mathcal{T})$ is called a $\mathcal{T}$-flat vector bundle [22]. In particular, if $\mathcal{T}$ is a CR structure then $(\mathcal{E}, D_\mathcal{T})$ is a CR vector bundle. Moreover, if $\mathcal{T}$ is the integrable bundle $T^{0,1}X$ of vectors of type $(0,1)$ then the $\mathcal{T}$-flat complex vector bundle $(\mathcal{E}, D_\mathcal{T})$ is a holomorphic bundle.

4.2 Field equations on the CR supertwistor space

A distribution $\mathcal{T}$ on $\mathcal{F}^{5|8}$. Consider the CR supertwistor space $\mathcal{F}^{5|8}$ and a distribution $\mathcal{T}$ generated by the vector fields $\bar{W}_1^\pm$, $\bar{W}_2^\pm$, $\bar{V}^i_\pm$ from the CR structure $\hat{\bar{\mathscr{D}}}$ and $\bar{W}_3^\pm$. This distribution is integrable since all conditions described in section 4.1 are satisfied, e.g. the only nonzero commutator is

$$[\bar{W}_2^\pm, \bar{W}_3^\pm] = \pm 2\gamma_\pm^2 \bar{W}_1^\pm \tag{4.9}$$

and therefore $\mathcal{T}$ is closed under the Lie bracket. Also,

$$\mathscr{V} := \mathcal{T} \cap \bar{\mathcal{T}} \tag{4.10}$$

is a real one-dimensional and hence integrable distribution. The vector fields $\bar{W}_3^\pm$ are a basis for $\mathscr{V}$ over the patches $\tilde{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$. Note that the mini-supertwistor space $\mathcal{P}^{2|4}$ is a subsupermanifold of $\mathcal{F}^{5|8}$ transversal to the leaves of $\mathscr{V} = \mathcal{T} \cap \bar{\mathcal{T}}$ and furthermore, $\mathcal{T}|_{\mathcal{P}^{2|4}} = \hat{\bar{\mathscr{D}}}$. Thus, we have an integrable distribution $\mathcal{T} = \hat{\bar{\mathscr{D}}} \oplus \mathscr{V}$ on the CR supertwistor space $\mathcal{F}^{5|8}$ and we will denote by $\mathcal{T}_b$ its bosonic part generated by the vector fields $\bar{W}_1^\pm$, $\bar{W}_2^\pm$ and $\bar{W}_3^\pm$,

$$\mathcal{T}_b := \mathrm{span}\{\bar{W}_1^\pm, \bar{W}_2^\pm, \bar{W}_3^\pm\}\,. \tag{4.11}$$

Holomorphic integral form. Let $\mathcal{E}$ be a trivial rank n complex vector bundle over $\mathcal{F}^{5|8}$ and $\mathcal{A}_\mathcal{T}$ a $\mathcal{T}$-connection one-form on $\mathcal{E}$ with $\mathcal{T} = \hat{\bar{\mathscr{D}}} \oplus \mathscr{V}$. Consider now the subspace $\mathscr{X}$ of $\mathcal{F}^{5|8}$ which is parametrized by the same bosonic coordinates but only the holomorphic fermionic coordinates of $\mathcal{F}^{5|8}$, i.e. on $\mathscr{X}$, all objects are holomorphic in $\eta_i^\pm$. As it was already noted in [1], the $\mathcal{N} = 4$ mini-supertwistor space is a Calabi-Yau supermanifold. In particular, this ensures the existence of a holomorphic volume form on $\mathcal{P}^{2|4}$. Moreover, $\mathcal{P}^{2|4} \subset \mathcal{F}^{5|8}$ and the pull-back $\tilde{\Omega}$ of this form is globally defined on[16] $\mathcal{F}^{5|8}$. Locally, on the patches $\tilde{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$, one obtains

$$\tilde{\Omega}|_{\tilde{\mathcal{V}}_\pm} = \pm \mathrm{d}w^1_\pm \wedge \mathrm{d}w^2_\pm \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm\,. \tag{4.12}$$

This well-defined integral form allows us to integrate on $\mathscr{X}$ by pairing it with elements from $\Omega^3_{\mathcal{T}_b}(\mathscr{X})$.

Action functional for phCS theory. Let us assume that $\mathcal{A}_\mathcal{T}$ contains no antiholomorphic fermionic components and does not depend on $\bar{\eta}_i^\pm$,

$$\bar{V}^i_\pm \lrcorner \mathcal{A}_\mathcal{T} = 0 \qquad \text{and} \qquad \bar{V}^i_\pm(\mathcal{A}_a^\pm) = 0 \qquad \text{with} \qquad \mathcal{A}_a^\pm := \bar{W}_a^\pm \lrcorner \mathcal{A}_\mathcal{T}\,, \tag{4.13}$$

[16]Recall that the integrable distribution $\hat{\bar{\mathscr{D}}}$ and its dual are related to the holomorphic tangent and cotangent bundles of $\mathcal{P}^{2|4}$.

i.e. $\mathcal{A}_{\mathcal{T}} \in \Omega^1_{\mathcal{T}_b}(\mathscr{X}, \operatorname{End}\mathcal{E})$. Now, we introduce a CS-type action functional

$$S_{\text{phCS}} = \int_{\mathscr{X}} \tilde{\Omega} \wedge \operatorname{tr}\left(\mathcal{A}_{\mathcal{T}} \wedge \mathrm{d}_{\mathcal{T}}\mathcal{A}_{\mathcal{T}} + \frac{2}{3}\mathcal{A}_{\mathcal{T}} \wedge \mathcal{A}_{\mathcal{T}} \wedge \mathcal{A}_{\mathcal{T}}\right), \tag{4.14}$$

where

$$\mathrm{d}_{\mathcal{T}}|_{\tilde{\mathcal{V}}_\pm} = \mathrm{d}\bar{w}^a_\pm \frac{\partial}{\partial \bar{w}^a_\pm} + \mathrm{d}\bar{\eta}^\pm_i \frac{\partial}{\partial \bar{\eta}^\pm_i} \tag{4.15}$$

is the $\mathcal{T}$-part of the exterior derivative d on $\mathcal{F}^{5|8}$.

Field equations of phCS theory. The action (4.14) leads to the CS-type field equations

$$\mathrm{d}_{\mathcal{T}}\mathcal{A}_{\mathcal{T}} + \mathcal{A}_{\mathcal{T}} \wedge \mathcal{A}_{\mathcal{T}} = 0, \tag{4.16}$$

which are the equations of motion of phCS theory. In the nonholonomic basis $\{\bar{W}^\pm_a, \bar{V}^i_\pm\}$ of the distribution $\mathcal{T}$ over $\tilde{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$, these equations read as

$$\bar{W}^\pm_1\mathcal{A}^\pm_2 - \bar{W}^\pm_2\mathcal{A}^\pm_1 + [\mathcal{A}^\pm_1, \mathcal{A}^\pm_2] = 0, \tag{4.17a}$$
$$\bar{W}^\pm_2\mathcal{A}^\pm_3 - \bar{W}^\pm_3\mathcal{A}^\pm_2 + [\mathcal{A}^\pm_2, \mathcal{A}^\pm_3] \mp 2\gamma^2_\pm\mathcal{A}^\pm_1 = 0, \tag{4.17b}$$
$$\bar{W}^\pm_1\mathcal{A}^\pm_3 - \bar{W}^\pm_3\mathcal{A}^\pm_1 + [\mathcal{A}^\pm_1, \mathcal{A}^\pm_3] = 0, \tag{4.17c}$$

where the components $\mathcal{A}^\pm_a$ have been defined in (4.13).

4.3 Equivalence to supersymmetric Bogomolny equations

Dependence on $\lambda_\pm$, $\bar{\lambda}_\pm$. Note that from (3.34b), it follows that

$$\bar{W}^+_1 = \lambda^2_+\bar{W}^-_1, \qquad \bar{W}^+_2 = -\bar{\lambda}^{-2}_+\bar{W}^-_2 \qquad \text{and} \qquad \gamma^{-1}_+\bar{W}^+_3 = \lambda_+\bar{\lambda}_+\left(\gamma^{-1}_-\bar{W}^-_3\right) \tag{4.18}$$

and therefore $\mathcal{A}^\pm_1$, $\mathcal{A}^\pm_2$ and $\gamma^{-1}_\pm\mathcal{A}^\pm_3$ take values in the bundles $\mathcal{O}(2)$, $\bar{\mathcal{O}}(-2)$ and $\mathcal{O}(1)\otimes\bar{\mathcal{O}}(1)$, respectively. Together with the definitions (4.13) of $\mathcal{A}^\pm_a$ and (3.34) of $\bar{W}^\pm_a$ as well as the fact that the $\eta^\pm_i$ are nilpotent and $\mathcal{O}(1)$-valued, this determines the dependence of $\mathcal{A}^\pm_a$ on $\eta^\pm_i$, $\lambda_\pm$ and $\bar{\lambda}_\pm$ to be

$$\mathcal{A}^\pm_1 = -\lambda^{\dot{\alpha}}_\pm\mathcal{B}^\pm_{\dot{\alpha}} \qquad \text{and} \qquad \mathcal{A}^\pm_3 = 2\gamma_\pm\hat{\lambda}^{\dot{\alpha}}_\pm\mathcal{B}^\pm_{\dot{\alpha}} \tag{4.19}$$

with the abbreviation

$$\begin{aligned}\mathcal{B}^\pm_{\dot{\alpha}} \;:=\; &\lambda^{\dot{\beta}}_\pm B_{\dot{\alpha}\dot{\beta}} + \mathrm{i}\eta^\pm_i\chi^i_{\dot{\alpha}} + \frac{1}{2!}\gamma_\pm\eta^\pm_i\eta^\pm_j\hat{\lambda}^{\dot{\beta}}_\pm\phi^{ij}_{\dot{\alpha}\dot{\beta}} + \frac{1}{3!}\gamma^2_\pm\eta^\pm_i\eta^\pm_j\eta^\pm_k\hat{\lambda}^{\dot{\beta}}_\pm\hat{\lambda}^{\dot{\gamma}}_\pm\tilde{\chi}^{ijk}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \;+\\ &+\frac{1}{4!}\gamma^3_\pm\eta^\pm_i\eta^\pm_j\eta^\pm_k\eta^\pm_l\hat{\lambda}^{\dot{\beta}}_\pm\hat{\lambda}^{\dot{\gamma}}_\pm\hat{\lambda}^{\dot{\delta}}_\pm G^{ijkl}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\end{aligned} \tag{4.20a}$$

and

$$\mathcal{A}^\pm_2 = \pm\left(\frac{1}{2!}\gamma^2_\pm\eta^\pm_i\eta^\pm_j\phi^{ij} + \frac{1}{3!}\gamma^3_\pm\eta^\pm_i\eta^\pm_j\eta^\pm_k\hat{\lambda}^{\dot{\alpha}}_\pm\tilde{\chi}^{ijk}_{\dot{\alpha}} + \frac{1}{4!}\gamma^4_\pm\eta^\pm_i\eta^\pm_j\eta^\pm_k\eta^\pm_l\hat{\lambda}^{\dot{\alpha}}_\pm\hat{\lambda}^{\dot{\beta}}_\pm G^{ijkl}_{\dot{\alpha}\dot{\beta}}\right), \tag{4.20b}$$

where $\lambda^{\dot{\alpha}}_\pm$, $\gamma_\pm$ and $\hat{\lambda}^{\dot{\alpha}}_\pm$ have been given in (2.22) and (3.35). The expansions (4.20) are defined up to gauge transformations generated by group-valued functions which may depend on $\lambda_\pm$ and $\bar{\lambda}_\pm$. In particular, it is assumed in this twistor correspondence that for solutions to (4.17), there exists a gauge in which terms of zeroth and first order in $\eta_i^\pm$ are absent in $\mathcal{A}_2^\pm$. In the Čech approach, this corresponds to the holomorphic triviality of the bundle $\tilde{\mathcal{E}}$ defined by such solutions when restricted to projective lines. Put differently, we consider a subset in the set of all solutions of phCS theory on $\mathcal{F}^{5|8}$, and we will always mean this subset when speaking of solutions to phCS theory. From the properties of $\mathcal{A}_a^\pm$ and $\eta_i^\pm$, it follows that the fields with an odd number of spinor indices are fermionic (odd) while those with an even number of spinor indices are bosonic (even). Moreover, due to the symmetry of the $\hat{\lambda}_\pm^{\dot\alpha}$ products and the antisymmetry of the $\eta_i^\pm$ products, all component fields are automatically symmetric in (some of) their spinor indices and antisymmetric in their Latin (R-symmetry) indices.

Supersymmetric Bogomolny equations. Note that in (4.20), all fields $B_{\dot\alpha\dot\beta}, \chi^i_{\dot\alpha}, \ldots$ depend only on the coordinates $(y^{\dot\alpha\dot\beta}) \in \mathbb{R}^3$. Substituting (4.20) into (4.17a) and (4.17b), we obtain the equations

$$\phi^{ij}_{\dot\alpha\dot\beta} = -\left(\partial_{(\dot\alpha\dot\beta)}\phi^{ij} + [B_{\dot\alpha\dot\beta}, \phi^{ij}]\right), \quad \tilde\chi^{ijk}_{\dot\alpha(\dot\beta\dot\gamma)} = -\frac{1}{2}\left(\partial_{(\dot\alpha(\dot\beta)}\tilde\chi^{ijk}_{\dot\gamma)} + [B_{\dot\alpha(\dot\beta}, \tilde\chi^{ijk}_{\dot\gamma)}]\right),$$
$$G^{ijkl}_{\dot\alpha(\dot\beta\dot\gamma\dot\delta)} = -\frac{1}{3}\left(\partial_{(\dot\alpha(\dot\beta)}G^{ijkl}_{\dot\gamma\dot\delta)} + [B_{\dot\alpha(\dot\beta}, G^{ijkl}_{\dot\gamma\dot\delta)}]\right) \tag{4.21}$$

showing that $\phi^{ij}_{\dot\alpha\dot\beta}$, $\tilde\chi^{ijk}_{\dot\alpha\dot\beta\dot\gamma}$ and $G^{ijkl}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ are composite fields, which do not describe independent degrees of freedom. Furthermore, the field $B_{\dot\alpha\dot\beta}$ can be decomposed into its symmetric part, denoted by $A_{\dot\alpha\dot\beta} = A_{(\dot\alpha\dot\beta)}$, and its antisymmetric part, proportional to Φ, such that

$$B_{\dot\alpha\dot\beta} = A_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi\,. \tag{4.22}$$

Hence, we have recovered the covariant derivative $D_{\dot\alpha\dot\beta} = \partial_{(\dot\alpha\dot\beta)} + A_{\dot\alpha\dot\beta}$ and the (scalar) Higgs field Φ. Defining

$$\tilde\chi_{i\dot\alpha} := \frac{1}{3!}\varepsilon_{ijkl}\tilde\chi^{jkl}_{\dot\alpha} \qquad \text{and} \qquad G_{\dot\alpha\dot\beta} := \frac{1}{4!}\varepsilon_{ijkl}G^{ijkl}_{\dot\alpha\dot\beta}\,, \tag{4.23}$$

we have thus obtained the supermultiplet in three dimensions consisting of the fields

$$A_{\dot\alpha\dot\beta}, \chi^i_{\dot\alpha}, \Phi, \phi^{ij}, \tilde\chi_{i\dot\alpha}, G_{\dot\alpha\dot\beta}\,. \tag{4.24}$$

The equations (4.17) together with the field expansions (4.20), the constraints (4.21) and the definitions (4.22) and (4.23) yield the following supersymmetric extension of the Bogomolny equations:

$$\begin{aligned}
f_{\dot\alpha\dot\beta} &= -\frac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi\,,\\
\varepsilon^{\dot\beta\dot\gamma}D_{\dot\alpha\dot\beta}\chi^i_{\dot\gamma} &= -\frac{\mathrm{i}}{2}[\Phi, \chi^i_{\dot\alpha}]\,,\\
\triangle\phi^{ij} &= -\frac{1}{4}[\Phi, [\phi^{ij}, \Phi]] + \varepsilon^{\dot\alpha\dot\beta}\{\chi^i_{\dot\alpha}, \chi^j_{\dot\beta}\}\,,\\
\varepsilon^{\dot\beta\dot\gamma}D_{\dot\alpha\dot\beta}\tilde\chi_{i\dot\gamma} &= -\frac{\mathrm{i}}{2}[\tilde\chi_{i\dot\alpha}, \Phi] + 2\mathrm{i}[\phi_{ij}, \chi^j_{\dot\alpha}]\,,\\
\varepsilon^{\dot\beta\dot\gamma}D_{\dot\alpha\dot\beta}G_{\dot\gamma\dot\delta} &= -\frac{\mathrm{i}}{2}[G_{\dot\alpha\dot\delta}, \Phi] + \mathrm{i}\{\chi^i_{\dot\alpha}, \tilde\chi_{i\dot\delta}\} - \frac{1}{2}[\phi_{ij}, D_{\dot\alpha\dot\delta}\phi^{ij}] + \frac{\mathrm{i}}{4}\varepsilon_{\dot\alpha\dot\delta}[\phi_{ij}, [\Phi, \phi^{ij}]]\,,
\end{aligned} \tag{4.25}$$

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which can also be derived from the equations (2.46) by demanding that all the fields in (2.46) are independent of the coordinate x^4. In (4.25), we have used the fact that we have a decomposition of the field strength in three dimensions according to

$$F_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = [D_{\dot\alpha\dot\beta}, D_{\dot\gamma\dot\delta}] \ =: \ \varepsilon_{\dot\beta\dot\delta} f_{\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma} f_{\dot\beta\dot\delta} \tag{4.26}$$

with $f_{\dot\alpha\dot\beta} = f_{\dot\beta\dot\alpha}$. We have also introduced the abbreviation $\triangle := \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\gamma\dot\delta} D_{\dot\alpha\dot\gamma} D_{\dot\beta\dot\delta}$.

Action functional in component fields. Note that (4.12) can be rewritten as

$$\tilde\Omega|_{\tilde{\mathcal{V}}_\pm} = \pm\Theta^1_\pm \wedge \Theta^2_\pm \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm \,, \tag{4.27}$$

where the one-forms $\Theta^{1,2}_\pm$ have been given in (3.36). Substituting this expression and the expansions (4.20) into the action (4.14), we arrive after a straightforward calculation at the action

$$\begin{aligned} S_{\mathrm{sB}} = \int \mathrm{d}^3x \ \mathrm{tr}\Big\{ & \varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma} G_{\dot\gamma\dot\delta}\left(f_{\dot\alpha\dot\beta} + \frac{\mathrm{i}}{2} D_{\dot\alpha\dot\beta}\Phi\right) + \mathrm{i}\varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma}\chi^i_{\dot\alpha} D_{\dot\delta\dot\beta}\tilde\chi_{i\dot\gamma} + \frac{1}{2}\phi_{ij}\triangle\phi^{ij} - \\ & - \frac{1}{2}\varepsilon^{\dot\alpha\dot\delta}\chi^i_{\dot\alpha}[\tilde\chi_{i\dot\delta}, \Phi] - \varepsilon^{\dot\alpha\dot\gamma}\phi_{ij}\{\chi^i_{\dot\alpha}, \chi^j_{\dot\gamma}\} + \frac{1}{8}[\phi_{ij}, \Phi][\phi^{ij}, \Phi]\Big\} \,, \end{aligned} \tag{4.28}$$

which yields the supersymmetric Bogomolny (sB) equations (4.25). In this expression, we have again used the shorthand notation $\phi_{ij} := \frac{1}{2!}\varepsilon_{ijkl}\phi^{kl}$.

4.4 Partially holomorphic CS theory in the Čech description

Equivalent $\mathcal{T}$-flat bundle. Our starting point in section 4.2 was to consider a trivial complex vector bundle $\mathcal{E}$ over $\mathcal{F}^{5|8}$ endowed with a $\mathcal{T}$-connection. Such a $\mathcal{T}$-connection $D_\mathcal{T} = \mathrm{d}_\mathcal{T} + \mathcal{A}_\mathcal{T}$ on $\mathcal{E}$ is flat if $\mathcal{A}_\mathcal{T}$ solves the equations (4.16), and then $(\mathcal{E}, f_{+-}{=}\,\mathbb{1}_n, D_\mathcal{T})$ is a $\mathcal{T}$-flat bundle in the Dolbeault description. After one turns to the Čech description of $\mathcal{T}$-flat bundles, the connection one-form $\mathcal{A}_\mathcal{T}$ disappears and all the information is hidden in a transition function. To achieve this, let us restrict a solution $\mathcal{A}_\mathcal{T}$ of the equations (4.16) to the patches $\tilde{\mathcal{V}}_+$ and $\tilde{\mathcal{V}}_-$ covering $\mathcal{F}^{5|8}$. Since $\mathcal{A}_\mathcal{T}$ is flat, it is given as a pure gauge configuration on each patch and we have

$$\mathcal{A}_\mathcal{T}|_{\tilde{\mathcal{V}}_\pm} = \psi_\pm \mathrm{d}_\mathcal{T}\psi_\pm^{-1} \,, \tag{4.29}$$

where the $\psi_\pm$ are smooth $\mathrm{GL}(n,\mathbb{C})$-valued superfunctions on $\tilde{\mathcal{V}}_\pm$ such that $\bar V^i_\pm \psi_\pm = 0$ (the existence of such a gauge was assumed in the formulation of phCS theory). Due to the triviality of the bundle $\mathcal{E}$, we have

$$\psi_+ \mathrm{d}_\mathcal{T}\psi_+^{-1} = \psi_- \mathrm{d}_\mathcal{T}\psi_-^{-1} \tag{4.30}$$

on the intersection $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$. From (4.30), one easily obtains

$$\mathrm{d}_\mathcal{T}(\psi_+^{-1}\psi_-) = 0 \,, \tag{4.31}$$

and we can define a $\mathcal{T}$-flat complex vector bundle $\tilde{\mathcal{E}} \to \mathcal{F}^{5|8}$ with the canonical flat $\mathcal{T}$-connection $\mathrm{d}_{\mathcal{T}}$ and the transition function

$$\tilde{f}_{+-} := \psi_+^{-1}\psi_- \,. \tag{4.32}$$

The bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent as smooth bundles but not as $\mathcal{T}$-flat bundles. However, we have an equivalence of the following data:

$$(\mathcal{E}, f_{+-} = \mathbb{1}_n, \mathcal{A}_{\mathcal{T}}) \;\sim\; (\tilde{\mathcal{E}}, \tilde{f}_{+-}, \tilde{\mathcal{A}}_{\mathcal{T}} = 0)\,. \tag{4.33}$$

Equivalent flat $\mathcal{T}$-connection. To improve our understanding of the $\mathcal{T}$-flatness of the bundle $\tilde{\mathcal{E}} \to \mathcal{F}^{5|8}$ with the transition function (4.32), we rewrite the conditions (4.31) as follows:

$$\bar{W}_1^+ \tilde{f}_{+-} = 0\,, \qquad \bar{W}_2^+ \tilde{f}_{+-} = 0\,, \qquad \bar{V}_+^i \tilde{f}_{+-} = 0\,, \tag{4.34a}$$
$$\bar{W}_3^+ \tilde{f}_{+-} = 0\,. \tag{4.34b}$$

Recall that $\mathcal{T} = \hat{\bar{\mathscr{D}}} \oplus \mathscr{V}$ and the vector fields appearing in (4.34a) generate the antiholomorphic distribution $\hat{\bar{\mathscr{D}}}$, which is a CR structure. In other words, the bundle $\tilde{\mathcal{E}}$ is holomorphic along the mini-supertwistor space $\mathcal{P}^{2|4} \subset \mathcal{F}^{5|8}$ and flat along the fibres of the projection $\pi_2 : \mathcal{F}^{5|8} \to \mathcal{P}^{2|4}$ as follows from (4.34b). Let us now additionally assume that $\tilde{\mathcal{E}}$ is holomorphically trivial[17] when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{F}^{5|8}$ given by (3.19). This extra assumption guarantees the existence of a gauge in which the component $\mathcal{A}_2^{\pm}$ of $\mathcal{A}_{\mathcal{T}}$ vanishes. Hence, there exist $\mathrm{GL}(n,\mathbb{C})$-valued functions $\hat{\psi}_\pm$ such that

$$\tilde{f}_{+-} = \psi_+^{-1}\psi_- = \hat{\psi}_+^{-1}\hat{\psi}_- \qquad \text{with} \qquad \bar{W}_2^{\pm}\hat{\psi}_\pm = 0 \tag{4.35}$$

and

$$g := \psi_+\hat{\psi}_+^{-1} = \psi_-\hat{\psi}_-^{-1} \tag{4.36}$$

is a matrix-valued function generating a gauge transformation

$$\psi_\pm \;\mapsto\; \hat{\psi}_\pm = g^{-1}\psi_\pm\,, \tag{4.37}$$

which acts on the gauge potential according to

$$\begin{aligned}
\mathcal{A}_1^{\pm} \;&\mapsto\; \hat{\mathcal{A}}_1^{\pm} = g^{-1}\mathcal{A}_1^{\pm}g + g^{-1}\bar{W}_1^{\pm}g = \hat{\psi}_\pm \bar{W}_1^{\pm}\hat{\psi}_\pm^{-1}\,,\\
\mathcal{A}_2^{\pm} \;&\mapsto\; \hat{\mathcal{A}}_2^{\pm} = g^{-1}\mathcal{A}_2^{\pm}g + g^{-1}\bar{W}_2^{\pm}g = \hat{\psi}_\pm \bar{W}_2^{\pm}\hat{\psi}_\pm^{-1} = 0\,,\\
\mathcal{A}_3^{\pm} \;&\mapsto\; \hat{\mathcal{A}}_3^{\pm} = g^{-1}\mathcal{A}_3^{\pm}g + g^{-1}\bar{W}_3^{\pm}g = \hat{\psi}_\pm \bar{W}_3^{\pm}\hat{\psi}_\pm^{-1}\,,\\
0 = \mathcal{A}_\pm^i := \psi_\pm \bar{V}_\pm^i \psi_\pm^{-1} \;&\mapsto\; \hat{\mathcal{A}}_\pm^i = g^{-1}\bar{V}_\pm^i g = \hat{\psi}_\pm \bar{V}_\pm^i \hat{\psi}_\pm^{-1}\,.
\end{aligned} \tag{4.38}$$

In this new gauge, one generically has $\hat{\mathcal{A}}_\pm^i \neq 0$.

[17]This assumption, which was already used in (4.20), is crucial in the twistor approach.

Linear systems. Note that (4.29) can be rewritten as the following linear system of differential equations:

$$
\begin{aligned}
(\bar{W}_a^{\pm} + \mathcal{A}_a^{\pm})\psi_{\pm} &= 0\,, \\
\bar{V}_{\pm}^{i}\psi_{\pm} &= 0\,.
\end{aligned} \tag{4.39}
$$

The compatibility conditions of this linear system are the equations (4.17). This means that for any solution $\mathcal{A}_a^{\pm}$ to (4.17), one can construct solutions $\psi_{\pm}$ to (4.39) and, conversely, for any given $\psi_{\pm}$ obtained via a splitting (4.32) of a transition function $\tilde{f}_{+-}$, one can construct a solution (4.29) to (4.17).

Similarly, the equations (4.38) can be rewritten as the gauge equivalent linear system

$$(\bar{W}_1^{\pm} + \hat{\mathcal{A}}_1^{\pm})\hat{\psi}_{\pm} = 0\,, \tag{4.40a}$$

$$\bar{W}_2^{\pm}\hat{\psi}_{\pm} = 0\,, \tag{4.40b}$$

$$(\bar{W}_3^{\pm} + \hat{\mathcal{A}}_3^{\pm})\hat{\psi}_{\pm} = 0\,, \tag{4.40c}$$

$$(\bar{V}_{\pm}^{i} + \hat{\mathcal{A}}_{\pm}^{i})\hat{\psi}_{\pm} = 0\,. \tag{4.40d}$$

Note that due to the holomorphicity of $\hat{\psi}_{\pm}$ in $\lambda_{\pm}$ and the condition $\hat{\mathcal{A}}_{\mathcal{I}}^{+} = \hat{\mathcal{A}}_{\mathcal{I}}^{-}$ on $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$, the components $\hat{\mathcal{A}}_1^{\pm}$, $\gamma_{\pm}^{-1}\hat{\mathcal{A}}_3^{\pm}$ and $\hat{\mathcal{A}}_{\pm}^{i}$ must take the form

$$\hat{\mathcal{A}}_1^{\pm} = -\lambda_{\pm}^{\dot\alpha}\lambda_{\pm}^{\dot\beta}\mathcal{B}_{\dot\alpha\dot\beta}\,, \qquad \gamma_{\pm}^{-1}\hat{\mathcal{A}}_3^{\pm} = 2\hat{\lambda}_{\pm}^{\dot\alpha}\lambda_{\pm}^{\dot\beta}\mathcal{B}_{\dot\alpha\dot\beta} \qquad \text{and} \qquad \hat{\mathcal{A}}_{\pm}^{i} \;=\; \lambda_{\pm}^{\dot\alpha}\mathcal{A}_{\dot\alpha}^{i}\,, \tag{4.41}$$

with λ-independent superfields $\mathcal{B}_{\dot\alpha\dot\beta} := \mathcal{A}_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi$ and $\mathcal{A}_{\dot\alpha}^{i}$. Defining the first-order differential operators $\nabla_{\dot\alpha\dot\beta} := \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}$ and $D_{\dot\alpha}^{i} = \frac{\partial}{\partial\eta_i^{\dot\alpha}} + \mathcal{A}_{\dot\alpha}^{i} =: \partial_{\dot\alpha}^{i} + \mathcal{A}_{\dot\alpha}^{i}$, we arrive at the following compatibility conditions of the linear system (4.40):

$$
\begin{aligned}
[\nabla_{\dot\alpha\dot\gamma}, \nabla_{\dot\beta\dot\delta}] + [\nabla_{\dot\alpha\dot\delta}, \nabla_{\dot\beta\dot\gamma}] = 0\,, \qquad [D_{\dot\alpha}^{i}, \nabla_{\dot\beta\dot\gamma}] + [D_{\dot\gamma}^{i}, \nabla_{\dot\beta\dot\alpha}] = 0\,, \\
\{D_{\dot\alpha}^{i}, D_{\dot\beta}^{j}\} + \{D_{\dot\beta}^{i}, D_{\dot\alpha}^{j}\} = 0\,.
\end{aligned} \tag{4.42}
$$

These equations also follow from (4.16) after substituting the expansions (4.41).

Superfield equations. The equations (4.42) can equivalently be rewritten as

$$[\nabla_{\dot\alpha\dot\gamma}, \nabla_{\dot\beta\dot\delta}] =: \varepsilon_{\dot\gamma\dot\delta}\Sigma_{\dot\alpha\dot\beta}\,, \qquad [D_{\dot\alpha}^{i}, \nabla_{\dot\beta\dot\gamma}] =: \mathrm{i}\varepsilon_{\dot\alpha\dot\gamma}\Sigma_{\dot\beta}^{i} \qquad \text{and} \qquad \{D_{\dot\alpha}^{i}, D_{\dot\beta}^{j}\} =: \varepsilon_{\dot\alpha\dot\beta}\Sigma^{ij}\,, \tag{4.43}$$

where $\Sigma_{\dot\alpha\dot\beta} = \Sigma_{\dot\beta\dot\alpha}$ and $\Sigma^{ij} = -\Sigma^{ji}$. Note that the first equation in (4.42) immediately shows that $f_{\dot\alpha\dot\beta} = -\frac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi$ and thus the contraction of the first equation of (4.43) with $\varepsilon^{\dot\gamma\dot\delta}$ gives $\Sigma_{\dot\alpha\dot\beta} = f_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi = 2f_{\dot\alpha\dot\beta}$. The graded Bianchi identities for the differential operators $\nabla_{\dot\alpha\dot\beta}$ and $D_{\dot\alpha}^{i}$ yield in a straightforward manner further field equations, which allow us to identify the superfields $\Sigma_{\dot\alpha}^{i}$ and Σ^{ij} with the spinors $\chi_{\dot\alpha}^{i}$ and the scalars ϕ^{ij}, respectively. Moreover, $\tilde{\chi}_{i\dot\alpha}$ is given by $\tilde{\chi}_{i\dot\alpha} := \frac{1}{3}\varepsilon_{ijkl}D_{\dot\alpha}^{j}\phi^{kl}$ and $G_{\dot\alpha\dot\beta}$ is defined by $G_{\dot\alpha\dot\beta} := -\frac{1}{4}D_{(\dot\alpha}^{i}\tilde{\chi}_{i\dot\beta)}$. Collecting the above information, one obtains the superfield equations for $\mathcal{A}_{\dot\alpha\dot\beta}$, $\chi_{\dot\alpha}^{i}$, Φ, ϕ^{ij}, $\tilde{\chi}_{i\dot\alpha}$ and $G_{\dot\alpha\dot\beta}$ which take the same form as (4.25) but with all the fields now being superfields.[18] Thus, the projection of the superfields onto the zeroth order components of their η-expansions gives (4.25).

[18] In the following, the zeroth order component in the η-expansion of a superfield (its body) is denoted by a "$\circ$".

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Recursion relations. To extract the physical field content from the superfields, we need their explicit expansions in powers of $\eta_i^{\dot\alpha}$. For this, we follow the literature [36] and impose the transversal gauge condition

$$\eta_i^{\dot\alpha}\mathcal{A}_{\dot\alpha}^i = 0\,, \tag{4.44}$$

which allows us to define the recursion operator

$$\mathcal{D} := \eta_i^{\dot\alpha} D_{\dot\alpha}^i = \eta_i^{\dot\alpha}\partial_{\dot\alpha}^i \qquad \text{with} \qquad \partial_{\dot\alpha}^i := \frac{\partial}{\partial\eta_i^{\dot\alpha}}\,. \tag{4.45}$$

Note that this gauge removes the superfluous gauge degrees of freedom associated with the fermionic coordinates. The constraint equations (4.43) together with the graded Bianchi identities yield the following recursion relations:

$$\begin{aligned}
(1+\mathcal{D})\mathcal{A}_{\dot\alpha}^i &= \varepsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\beta}\phi^{ij}\,,\\
\mathcal{D}\mathcal{B}_{\dot\alpha\dot\beta} &= -\mathrm{i}\varepsilon_{\dot\beta\dot\gamma}\eta_i^{\dot\gamma}\chi_{\dot\alpha}^i\,,\\
\mathcal{D}\chi_{\dot\alpha}^i &= \mathrm{i}\eta_j^{\dot\beta}\nabla_{\dot\alpha\dot\beta}\phi^{ij}\,,\\
\mathcal{D}\phi_{ij} &= -\eta_{[i}^{\dot\alpha}\tilde\chi_{j]\dot\alpha}\,,\\
\mathcal{D}\tilde\chi_{i\dot\alpha} &= -\eta_i^{\dot\beta}G_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\beta}[\phi^{jk},\phi_{ki}]\,,\\
\mathcal{D}G_{\dot\alpha\dot\beta} &= \eta_i^{\dot\gamma}\varepsilon_{\dot\gamma(\dot\alpha}[\tilde\chi_{j\dot\beta)},\phi^{ij}]\,.
\end{aligned} \tag{4.46}$$

These equations define order by order all the superfield expansions.

The explicit derivation of the expansions of the fields $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}_{\dot\alpha}^i$ is performed in appendix A. Here, we just quote the result which we will need later on:

$$\begin{aligned}
\mathcal{B}_{\dot\alpha\dot\beta} &= \mathring{\mathcal{B}}_{\dot\alpha\dot\beta} - \mathrm{i}\varepsilon_{\dot\beta\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\mathring{\chi}_{\dot\alpha}^{j_1} + \frac{1}{2!}\varepsilon_{\dot\beta\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\eta_{j_2}^{\dot\gamma_2}\nabla_{\dot\alpha\dot\gamma_2}\mathring{\phi}^{j_1j_2} - \frac{1}{2\cdot 3!}\varepsilon_{\dot\beta\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\eta_{j_2}^{\dot\gamma_2}\eta_{j_3}^{\dot\gamma_3}\varepsilon^{j_1j_2j_3k}\times\\
&\quad\times\nabla_{\dot\alpha\dot\gamma_2}\mathring{\tilde\chi}_{k\dot\gamma_3} - \frac{1}{4!}\varepsilon_{\dot\beta\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\eta_{j_2}^{\dot\gamma_2}\eta_{j_3}^{\dot\gamma_3}\eta_{j_4}^{\dot\gamma_4}\varepsilon^{j_1j_2j_3j_4}\nabla_{\dot\alpha\dot\gamma_2}\mathring{G}_{\dot\gamma_3\dot\gamma_4} + \cdots
\end{aligned} \tag{4.47a}$$

$$\begin{aligned}
\mathcal{A}_{\dot\alpha}^i &= \frac{1}{2!}\varepsilon_{\dot\alpha\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\mathring{\phi}^{ij_1} - \frac{1}{3!}\varepsilon_{\dot\alpha\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\eta_{j_2}^{\dot\gamma_2}\varepsilon^{ij_1j_2k}\mathring{\tilde\chi}_{k\dot\gamma_2} +\\
&\quad+\frac{3}{2\cdot 4!}\varepsilon_{\dot\alpha\dot\gamma_1}\eta_{j_1}^{\dot\gamma_1}\eta_{j_2}^{\dot\gamma_2}\eta_{j_3}^{\dot\gamma_3}\varepsilon^{ij_1j_2j_3}\mathring{G}_{\dot\gamma_2\dot\gamma_3} + \cdots\,.
\end{aligned} \tag{4.47b}$$

The equations (4.43) are satisfied for these expansions, if the supersymmetric Bogomolny equations (4.25) hold for the physical fields[19] appearing in the above expansions and vice versa.

Bijection between moduli spaces. Summarizing the discussion of this section, we have described a bijection between the moduli space $\mathcal{M}_{\rm phCS}$ of solutions[20] to the field equations (4.17) of phCS theory and the moduli space $\mathcal{M}_{\rm sB}$ of solutions to the supersymmetric Bogomolny equations (4.25),

$$\mathcal{M}_{\rm phCS} \;\longleftrightarrow\; \mathcal{M}_{\rm sB}\,. \tag{4.48}$$

[19] The fields $\mathring{G}_{\dot\alpha\dot\beta}, \mathring{\Phi}$, etc. are the same as the field $G_{\dot\alpha\dot\beta}, \Phi$, etc. in equations (4.25), but in the present section, we need to clearly distinguish between superfields and their bodies.

[20] Recall that we always consider only a subset of the full solution space as discussed in section 4.3.

The moduli spaces are obtained from the respective solution spaces by taking the quotient with respect to the action of the corresponding groups of gauge transformations. We also have shown that there is a one-to-one correspondence between gauge equivalence classes of solutions $\mathcal{A}_{\mathcal{T}}$ to the phCS field equations (4.16) and equivalence classes of topologically trivial $\mathcal{T}$-flat vector bundles $\tilde{\mathcal{E}}$ over the CR supertwistor space $\mathcal{F}^{5|8}$. In other words, we have demonstrated an equivalence of the Dolbeault and the Čech descriptions of the moduli space of $\mathcal{T}$-flat bundles.

5. Holomorphic BF theory on the mini-supertwistor space

In the preceding section, we have defined a theory on the CR supertwistor space $\mathcal{F}^{5|8}$ entering into the double fibration (3.20) which we called partially holomorphic Chern-Simons theory. We have shown that this theory is equivalent to a supersymmetric Bogomolny-type Yang-Mills-Higgs theory in three euclidean dimensions. The purpose of this section is to show, that one can also introduce a theory (including an action functional) on the mini-supertwistor space $\mathcal{P}^{2|4}$, which is equivalent to phCS theory on $\mathcal{F}^{5|8}$. Thus, one can define at each level of the double fibration (3.20) a theory accompanied by a proper action functional and, moreover, these three theories are all equivalent.

5.1 Field equations of hBF theory on $\mathcal{P}^{2|4}$

Consider the mini-supertwistor space $\mathcal{P}^{2|4}$. Let E be a trivial rank n complex vector bundle over $\mathcal{P}^{2|4}$ with a connection one-form $\mathcal{A}$. Assume that its $(0,1)$ part $\mathcal{A}^{0,1}$ contains no antiholomorphic fermionic components and does not depend on $\bar{\eta}_i^{\pm}$, i.e. $\bar{V}^i_{\pm} \lrcorner \mathcal{A}^{0,1} = 0$ and $\bar{V}^i_{\pm}(\partial_{\bar{w}^{1,2}_{\pm}} \lrcorner \mathcal{A}^{0,1}) = 0$. Recall that on $\mathcal{P}^{2|4}$, we have a holomorphic volume form Ω which is locally given by (3.15). Hence, we can define a holomorphic BF (hBF) type theory (cf. [17 − 19]) on $\mathcal{P}^{2|4}$ with the action

$$S_{\text{hBF}} = \int_{\mathscr{Y}} \Omega \wedge \operatorname{tr}\{B(\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1})\} = \int_{\mathscr{Y}} \Omega \wedge \operatorname{tr}\{B\mathcal{F}^{0,2}\}\,, \tag{5.1}$$

where B is a scalar field in the adjoint representation of the group $\mathrm{GL}(n,\mathbb{C})$, $\bar{\partial}$ is the antiholomorphic part of the exterior derivative on $\mathcal{P}^{2|4}$ and $\mathcal{F}^{0,2}$ the $(0,2)$ part of the curvature two-form. The space $\mathscr{Y}$ is the subsupermanifold of $\mathcal{P}^{2|4}$ constrained by $\bar{\eta}_i^{\pm} = 0$. In fact, $\mathscr{Y}$ is the worldvolume of a stack of n not quite space-filling D3-branes, as discussed in the introduction.

The equations of motion following from the action functional (5.1) are

$$\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0\,, \tag{5.2a}$$

$$\bar{\partial}B + [\mathcal{A}^{0,1}, B] = 0\,. \tag{5.2b}$$

These equations as well as the lagrangian in (5.1) can be obtained from the equations (4.16) and the lagrangian in (4.14), respectively, by imposing the condition $\partial_{\bar{w}^3_{\pm}} \mathcal{A}_{\bar{w}^a_{\pm}} = 0$ and identifying

$$\mathcal{A}^{0,1}|_{\hat{\mathcal{V}}_{\pm}} = \mathrm{d}\bar{w}^1_{\pm}\mathcal{A}_{\bar{w}^1_{\pm}} + \mathrm{d}\bar{w}^2_{\pm}\mathcal{A}_{\bar{w}^2_{\pm}} \qquad \text{and} \qquad B^{\pm} := B|_{\hat{\mathcal{V}}_{\pm}} = \mathcal{A}_{\bar{w}^3_{\pm}}\,. \tag{5.3}$$

Note that $\mathcal{A}_{\bar{w}^3_\pm}$ behaves on $\mathcal{P}^{2|4}$ as a scalar. Thus, (5.2) can be obtained from (4.16) by demanding invariance of all fields under the action of the group $\mathscr{G}'$ from section 3.4 such that $\mathcal{P}^{2|4} \cong \mathcal{F}^{5|8}/\mathscr{G}'$.

5.2 Čech description

When restricted to the patches $\hat{\mathcal{V}}_\pm$, the equations (5.2) can be solved by

$$\mathcal{A}^{0,1}|_{\hat{\mathcal{V}}_\pm} = \tilde{\psi}_\pm \bar{\partial} \tilde{\psi}_\pm^{-1} \qquad \text{and} \qquad B^\pm = \tilde{\psi}_\pm B_0^\pm \tilde{\psi}_\pm^{-1}\,, \tag{5.4}$$

where $B_0^\pm$ is a holomorphic $\mathrm{gl}(n,\mathbb{C})$-valued function on $\hat{\mathcal{V}}_\pm$,

$$\bar{\partial} B_0^\pm = 0\,. \tag{5.5}$$

On the intersection $\hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}_-$, we have the gluing conditions

$$\tilde{\psi}_+ \bar{\partial} \tilde{\psi}_+^{-1} = \tilde{\psi}_- \bar{\partial} \tilde{\psi}_-^{-1} \qquad \text{and} \qquad \tilde{\psi}_+ B_0^+ \tilde{\psi}_+^{-1} = \tilde{\psi}_- B_0^- \tilde{\psi}_-^{-1} \tag{5.6}$$

as E is a trivial bundle. From (5.6), we learn that

$$\tilde{f}_{+-} := \tilde{\psi}_+^{-1} \tilde{\psi}_- \tag{5.7}$$

can be identified with the holomorphic transition function of a bundle $\tilde{E}$ with the canonical holomorphic structure $\bar{\partial}$, and

$$B_0^+ = \tilde{f}_{+-} B_0^- \tilde{f}_{+-}^{-1} \tag{5.8}$$

is a global holomorphic section of the bundle $\mathrm{End}\,\tilde{E}$, i.e. $B_0 \in H^0(\mathcal{P}^{2|4}, \mathrm{End}\,\tilde{E})$ and $B \in H^0(\mathcal{P}^{2|4}, \mathrm{End}\,E)$. Note that the pull-back $\pi_2^* \tilde{E}$ of the bundle $\tilde{E}$ to the space $\mathcal{F}^{5|8}$ can be identified with the bundle $\tilde{\mathcal{E}}$,

$$\tilde{\mathcal{E}} = \pi_2^* \tilde{E}\,, \tag{5.9}$$

with the transition function $\tilde{f}_{+-} = \psi_+^{-1}\psi_- = \tilde{\psi}_+^{-1}\tilde{\psi}_-$. Recall that the transition functions of the bundle $\tilde{\mathcal{E}}$ do not depend on $w^3_\pm$ and therefore they can always be considered as the pull-backs of transition functions of a bundle $\tilde{E}$ over $\mathcal{P}^{2|4}$.

5.3 Moduli space

By construction, $B = \{B^\pm\}$ is a $\mathrm{gl}(n,\mathbb{C})$-valued function generating trivial infinitesimal gauge transformations of $\mathcal{A}^{0,1}$ and therefore it does not contain any physical degrees of freedom. Remember that solutions to the equations (5.2a) are defined up to gauge transformations

$$\mathcal{A}^{0,1} \;\mapsto\; \tilde{\mathcal{A}}^{0,1} = g\mathcal{A}^{0,1}g^{-1} + g\bar{\partial}g^{-1} \tag{5.10}$$

generated by smooth $\mathrm{GL}(n,\mathbb{C})$-valued functions g on $\mathcal{P}^{2|4}$. The transformations (5.10) do not change the holomorphic structure $\bar{\partial}_\mathcal{A}$ on the bundle E and the two $(0,1)$-connections in (5.10) are considered as equivalent. On infinitesimal level, the transformations (5.10) take the form

$$\delta\mathcal{A}^{0,1} = \bar{\partial}B + [\mathcal{A}^{0,1}, B] \tag{5.11}$$

with $B \in H^0(\mathcal{P}^{2|4}, \operatorname{End} E)$ and such a field B solving (5.2b) generates holomorphic transformations such that $\delta\mathcal{A}^{0,1} = 0$. Their finite version is

$$\tilde{\mathcal{A}}^{0,1} = g\mathcal{A}^{0,1}g^{-1} + g\bar{\partial}g^{-1} = \mathcal{A}^{0,1}\,, \tag{5.12}$$

and for a gauge potential $\mathcal{A}^{0,1}$ given by (5.4), such a g takes the form

$$g_\pm = \tilde{\psi}_\pm \mathrm{e}^{B_0^\pm}\tilde{\psi}_\pm^{-1} \qquad \text{with} \qquad g_+ = g_- \quad \text{on} \quad \hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}_-\,. \tag{5.13}$$

Thus, the hBF theory given by the action (5.1) and the field equations (5.2) describes holomorphic structures on the bundle $E \to \mathcal{P}^{2|4}$ and its moduli space $\mathcal{M}_{\mathrm{hBF}}$ is bijective to the moduli space of holomorphic bundles $\tilde{E}$ defined by the transition functions (5.7). Furthermore, this moduli space is bijective to the moduli space $\mathcal{M}_{\mathrm{phCS}}$ of $\mathcal{T}$-flat bundles $\tilde{\mathcal{E}}$ over the CR supertwistor space $\mathcal{F}^{5|8}$. Summarizing our above discussion, we have established the diagram

$$\begin{tikzcd} & \text{phCS theory on } \mathcal{F}^{5|8} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{hBF theory on } \mathcal{P}^{2|4} \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{supersymmetric}\\ \text{Bogomolny model on } \mathbb{R}^3\end{array} \end{tikzcd} \tag{5.14}$$

describing equivalent theories defined on different spaces. Here it is again implied that the appropriate subsets of the solution spaces to phCS and hBF theories are considered as discussed in section 4.3 and above.

6. Supersymmetric Bogomolny equations with massive fields

In an interesting recent paper [1], a twistor string theory corresponding to a certain massive super Yang-Mills theory in three dimensions was developed. It was argued, that the mass terms in this theory arise from coupling the R-symmetry current to a constant background field when performing the dimensional reduction. In this section, we want to study the analogous construction for the supersymmetric Bogomolny model which we discussed in the previous sections. We focus on the geometric origin of the additional mass terms by discussing the associated twistor description. More explicitly, we establish a correspondence between holomorphic bundles over the deformed mini-supertwistor space introduced in [1] and solutions to massive supersymmetric Bogomolny equations in three dimensions.

6.1 Mini-supertwistor and CR supertwistor spaces as vector bundles

$\mathcal{P}^{2|4}$ as a supervector bundle. We start from the observation that the mini-supertwistor space $\mathcal{P}^{2|4}$ can be considered as the total space of a rank $0|4$ holomorphic supervector bundle[21] over the mini-twistor space $\mathcal{P}^2$, i.e. a holomorphic vector bundle with Graßmann odd fibres. More explicitly, we have

$$\mathcal{P}^{2|4} = \mathcal{O}(2) \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^4 \tag{6.1}$$

[21] A complex (real) supervector bundle of rank $p|q$ is a vector bundle, whose typical fibre is the superspace $\mathbb{C}^{p|q}$ ($\mathbb{R}^{p|q}$). We also refer to such a supervector bundle simply as a vector bundle of rank $p|q$.

together with a holomorphic projection

$$\mathcal{P}^{2|4} \;\to\; \mathcal{P}^2 \,. \tag{6.2}$$

Recall that the mini-twistor space $\mathcal{P}^2$ is covered by two patches $\mathcal{V}_\pm$ with coordinates $w_\pm^1$ and $w_\pm^2 = \lambda_\pm$. The additional fibre coordinates in the supervector bundle $\mathcal{P}^{2|4}$ over $\mathcal{P}^2$ are the Graßmann variables $\eta_i^\pm$. For later convenience, we rearrange them into the vector $\eta^\pm = (\eta_i^\pm) \in \mathbb{C}^{0|4}$. On $\mathcal{V}_+ \cap \mathcal{V}_-$, we have the relation

$$\eta^+ = \varphi_{+-}\eta^- \tag{6.3}$$

with the transition function

$$\varphi_{+-} = w_+^2(\delta_i{}^j) = w_+^2 \mathbb{1}_4 \,. \tag{6.4}$$

$\mathcal{F}^{5|8}$ as a supervector bundle. The CR supertwistor space $\mathcal{F}^{5|8}$ is a CR supervector bundle[22] over the CR twistor space $\mathcal{F}^5 \cong \mathbb{R}^3 \times S^2$,

$$\mathcal{F}^{5|8} \;\to\; \mathcal{F}^5 \,, \tag{6.5}$$

with complex coordinates $\eta_i^\pm$ on the fibres $\mathbb{C}^{0|4}$ over the patches $\mathcal{V}'_\pm$ covering $\mathcal{F}^5$. Recall that we have the double fibration

$$\begin{tikzcd} & \mathcal{F}^5 \arrow[dl, "\nu_2"'] \arrow[dr, "\nu_1"] & \\ \mathcal{P}^2 & & \mathbb{R}^3 \end{tikzcd} \tag{6.6}$$

in the purely bosonic case and the transition function of the supervector bundle (6.5) can be identified with

$$\nu_2^*\varphi_{+-} = \lambda_+(\delta_i{}^j) = \lambda_+ \mathbb{1}_4 \,, \tag{6.7}$$

i.e., we have the same transformation (6.3) relating[23] η^+ to η^- on $\mathcal{V}'_+ \cap \mathcal{V}'_-$.

Combined double fibrations. Finally, note that $\mathbb{C}^{0|4} \cong \mathbb{R}^{0|8}$ and the superspace $\mathbb{R}^{3|8} \cong \mathbb{R}^3 \times \mathbb{R}^{0|8} \cong \mathbb{R}^3 \times \mathbb{C}^{0|4}$ can be considered as a trivial supervector bundle over $\mathbb{R}^3$ with the canonical projection

$$\mathbb{R}^{3|8} \;\to\; \mathbb{R}^3 \,. \tag{6.8}$$

Thus, we arrive at the diagram

$$\begin{tikzcd} & \mathcal{F}^{5|8} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] \arrow[dd] & \\ \mathcal{P}^{2|4} \arrow[dd] & & \mathbb{R}^{3|8} \arrow[dd] \\ & \mathcal{F}^5 \arrow[dl, "\nu_2"'] \arrow[dr, "\nu_1"] & \\ \mathcal{P}^2 & & \mathbb{R}^3 \end{tikzcd} \tag{6.9}$$

combining the double fibrations (3.20) and (6.6).

[22] i.e., it has a transition function annihilated by the vector fields $\partial_{\bar{w}_\pm^1}$, $\partial_{\bar{w}_\pm^2}$ from the distribution $\bar{\mathscr{D}}$ on $\mathcal{F}^5$

[23] Note that our notation often does not distinguish between objects on $\mathcal{P}^2$ and their pull-backs to $\mathcal{F}^5$.

6.2 Deformed mini-supertwistor and CR supertwistor spaces

$\mathcal{P}^{2|4}_M$ as a supervector bundle. Let us define a holomorphic supervector bundle

$$\mathcal{P}^{2|4}_M \;\to\; \mathcal{P}^2 \tag{6.10}$$

with complex coordinates $\tilde\eta^\pm = (\tilde\eta^\pm_i) \in \mathbb{C}^{0|4}$ on the fibres over $\mathcal{V}_\pm \subset \mathcal{P}^2$ which are related by the transition function

$$\tilde\varphi_{+-} = w^2_+ \mathrm{e}^{\frac{w^1_+}{w^2_+}M} \tag{6.11}$$

on the intersection $\mathcal{V}_+ \cap \mathcal{V}_-$, i.e.

$$\tilde\eta^+ = \tilde\varphi_{+-}\tilde\eta^- \,. \tag{6.12}$$

For reasons which will become more transparent in the later discussion, we demand that M is traceless and hermitean. The matrix M will eventually be the mass matrix of the fermions in three dimensions.

This supermanifold $\mathcal{P}^{2|4}_M$ was introduced in [1] as the target space of twistor string theories[24] which correspond, as proposed in this paper, to a supersymmetric Yang-Mills theory in three dimensions with massive spinors and both massive and massless scalar fields for hermitean matrices M. In the following, we provide a twistorial derivation of analogous mass terms in our supersymmetric Bogomolny model and explain their geometric origin.

$\mathcal{F}^{5|8}_M$ as a supervector bundle. Consider the rank $0|4$ holomorphic supervector bundle (6.10) and its pull-back

$$\mathcal{F}^{5|8}_M := \nu_2^*\mathcal{P}^{2|4}_M \;\to\; \mathcal{F}^5 \tag{6.13}$$

to the space $\mathcal{F}^5$ from the double fibration (6.6). Note that the supervector bundle $\mathcal{F}^{5|8}_M \to \mathcal{F}^5$ is smoothly equivalent to the supervector bundle $\mathcal{F}^{5|8} \to \mathcal{F}^5$ since in the coordinates $(y^{\dot\alpha\dot\beta}, \lambda_\pm, \bar\lambda_\pm) = (y, \bar y, x^3, \lambda_\pm, \bar\lambda_\pm)$ on $\mathcal{F}^5$, the pulled-back transition function $\nu_2^*\tilde\varphi_{+-}$ can be split

$$\nu_2^*\tilde\varphi_{+-} = \lambda_+ \mathrm{e}^{\frac{1}{\lambda_+}y^{\dot\alpha\dot\beta}\lambda^+_{\dot\alpha}\lambda^+_{\dot\beta}M} = \varphi_+(\lambda_+\mathbb{1}_4)\varphi_-^{-1} \;\sim\; \lambda_+\mathbb{1}_4\,. \tag{6.14}$$

Here,

$$\varphi_+ = \mathrm{e}^{-(x^3+\lambda_+\bar y)M} = \mathrm{e}^{\lambda^+_{\dot\alpha}y^{\dot\alpha\dot2}M} \qquad \text{and} \qquad \varphi_- = \mathrm{e}^{(x^3-\lambda_- y)M} = \mathrm{e}^{-\lambda^-_{\dot\alpha}y^{\dot\alpha\dot1}M} \tag{6.15}$$

are matrix-valued functions well-defined on the patches $\mathcal{V}'_+$ and $\mathcal{V}'_-$, respectively. Remember that $\tilde\eta^+_i$ and $\tilde\eta^-_i$ are related by (6.12) and their pull-backs to $\mathcal{F}^5$ (which we denote again by the same letter) are related by the transition function (6.14). Therefore, we have

$$(\varphi_+^{-1}\tilde\eta^+) = \lambda_+(\varphi_-^{-1}\tilde\eta^-)\,. \tag{6.16}$$

From this, we conclude that

$$\tilde\eta^+ = \varphi_+\eta^+ = \mathrm{e}^{\lambda^+_{\dot\alpha}y^{\dot\alpha\dot2}M}\eta^+ \qquad \text{and} \qquad \tilde\eta^- = \varphi_-\eta^- = \mathrm{e}^{-\lambda^-_{\dot\alpha}y^{\dot\alpha\dot1}M}\eta^-\,, \tag{6.17}$$

where $\eta^\pm = (\eta^\pm_i)$ are the fibre coordinates of the bundle (6.5) related by (6.3) on the intersection $\mathcal{V}'_+ \cap \mathcal{V}'_-$.

[24] For this, $\mathcal{P}^{2|4}_M$ has to be a Calabi-Yau supermanifold, which is the reason underlying the above restriction to $\operatorname{tr} M = 0$, as we will discuss.

The fibration $\mathcal{F}^{5|8}_M \to \mathcal{F}^5$ in the Dolbeault picture. It follows from (6.17) that

$$\bar{W}^\pm_1 \tilde{\eta}^\pm_i = 0\,, \qquad \bar{W}^\pm_2 \tilde{\eta}^\pm_i = 0 \qquad \text{and} \qquad \bar{W}^\pm_3 \tilde{\eta}^\pm_i + M_i{}^j \tilde{\eta}^\pm_j = 0\,. \tag{6.18}$$

Recall that the vector fields $\bar{W}^\pm_a$ generate an integrable (bosonic) distribution $\mathcal{T}_b = \mathrm{span}\{\bar{W}^\pm_a\}$ together with the operator

$$\mathrm{d}_{\mathcal{T}_b}|_{\mathcal{V}'_\pm} = \mathrm{d}\bar{w}^a_\pm \frac{\partial}{\partial \bar{w}^a_\pm}\,, \tag{6.19}$$

which annihilates the transition function (6.14) of the bundle (6.13). Due to formulæ (6.14) and (6.18), the supervector bundle $\mathcal{F}^{5|8}_M$ with canonical $\mathcal{T}_b$-flat connection $\mathrm{d}_{\mathcal{T}_b}$ is diffeomorphic to the supervector bundle $\mathcal{F}^{5|8}$ with the $\mathcal{T}_b$-flat connection $\mathrm{d}_{\mathcal{T}_b} + \hat{\mathcal{A}}_{\mathcal{T}_b}$ the components $\hat{\mathcal{A}}^\pm_a = \bar{W}^\pm_a \lrcorner \hat{\mathcal{A}}_{\mathcal{T}_b}$ of which are given by

$$\hat{\mathcal{A}}^\pm_1 = 0\,, \qquad \hat{\mathcal{A}}^\pm_2 = 0 \qquad \text{and} \qquad \hat{\mathcal{A}}^\pm_3 = M\,. \tag{6.20}$$

In other words, we have an equivalence of the following data:

$$(\mathcal{F}^{5|8}_M, \tilde{\varphi}_{+-}, \mathrm{d}_{\mathcal{T}_b}) \;\sim\; (\mathcal{F}^{5|8}, \varphi_{+-} = \lambda_+ \mathbb{1}_4, \mathrm{d}_{\mathcal{T}_b} + \hat{\mathcal{A}}_{\mathcal{T}_b})\,. \tag{6.21}$$

By construction, the connection one-form $\hat{\mathcal{A}}_{\mathcal{T}_b}$, given explicitly in (6.20), is a solution to the field equations

$$\mathrm{d}_{\mathcal{T}_b} \hat{\mathcal{A}}_{\mathcal{T}_b} + \hat{\mathcal{A}}_{\mathcal{T}_b} \wedge \hat{\mathcal{A}}_{\mathcal{T}_b} = 0 \tag{6.22}$$

of phCS theory on $\mathcal{F}^5$, which are equivalent via the arguments of section 4.3 to the Bogomolny equations on $\mathbb{R}^3$. Due to this correspondence, (6.20) is equivalent to a solution of the Bogomolny equations with vanishing Yang-Mills gauge potential $a_{\dot\alpha\dot\beta}$ and constant Higgs field

$$\phi = (\phi_i{}^j) = -\mathrm{i}(M_i{}^j)\,, \tag{6.23}$$

which takes values in the Lie algebra su(4) of the R-symmetry group SU(4). Thus, the data (6.21) are equivalent to the trivial supervector bundle (6.8) together with the differential operators

$$\nabla_{\dot\alpha\dot\beta} = \partial_{(\dot\alpha\dot\beta)} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta} M \tag{6.24}$$

encoding the information about the matrix M, i.e.

$$(\mathcal{F}^{5|8}_M, \tilde{\varphi}_{+-}, \mathrm{d}_{\mathcal{T}_b}) \;\sim\; \left(\mathcal{F}^{5|8}, \varphi_{+-}, \mathrm{d}_{\mathcal{T}_b} + \hat{\mathcal{A}}_{\mathcal{T}_b}\right) \;\sim\; \left(\mathbb{R}^{3|8}, \nabla_{\dot\alpha\dot\beta} = \partial_{(\dot\alpha\dot\beta)} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta} M\right). \tag{6.25}$$

Note that the gauge potential $A_{\dot\alpha\dot\beta}$ corresponding to[25] $A_r \in \mathrm{u}(n)$ in a different basis and the Higgs fields $\Phi \in \mathrm{u}(n)$ considered in section 4 can be combined with $a_{\dot\alpha\dot\beta}$ and ϕ into the fields

$$A_{\dot\alpha\dot\beta} \otimes \mathbb{1}_4 + \mathbb{1}_n \otimes a_{\dot\alpha\dot\beta} \qquad \text{and} \qquad \Phi \otimes \mathbb{1}_4 + \mathbb{1}_n \otimes \phi \tag{6.26}$$

acting on the tensor product $V_{\mathrm{U}(n)} \otimes V_{\mathrm{SU}(4)}$ of the (adjoint) representation space $V_{\mathrm{U}(n)}$ of the gauge group and the representation space $V_{\mathrm{SU}(4)}$ of the R-symmetry group.

[25] Here, A_r with $r = 1, 2, 3$ are the components of the ordinary gauge potential in three dimensions.

The fibration $\mathcal{P}^{2|4}_M \to \mathcal{P}^2$ in the Dolbeault picture. For completeness, we note that the deformed complex supervector bundle $\mathcal{P}^{2|4}_M \to \mathcal{P}^2$ with the transition function $\tilde{\varphi}_{+-}$ from (6.11) and the holomorphic structure

$$\bar{\partial}_b|_{\mathcal{V}_\pm} = \mathrm{d}\bar{w}^1_\pm \frac{\partial}{\partial \bar{w}^1_\pm} + \mathrm{d}\bar{w}^2_\pm \frac{\partial}{\partial \bar{w}^2_\pm} \tag{6.27}$$

is smoothly equivalent to the bundle $\mathcal{P}^{2|4} \to \mathcal{P}^2$ with the transition function φ_{+-} from (6.4) and the holomorphic structure defined by the fields $\hat{\mathcal{A}}^{0,1}$ and $\hat{B}$ with the components[26]

$$\hat{\mathcal{A}}_{\bar{w}^1_\pm} = 0\,, \qquad \hat{\mathcal{A}}_{\bar{w}^2_\pm} = \mp \frac{w^1_\pm}{(1+w^2_\pm \bar{w}^2_\pm)^2} M \qquad \text{and} \qquad \hat{B}_\pm = \hat{\mathcal{A}}_{\bar{w}^3_\pm} = M\,. \tag{6.28}$$

The fields $\hat{\mathcal{A}}^{0,1}$ and $\hat{B}$ obviously satisfy the field equations

$$\bar{\partial}_b \hat{\mathcal{A}}^{0,1} + \hat{\mathcal{A}}^{0,1} \wedge \hat{\mathcal{A}}^{0,1} = 0 \qquad \text{and} \qquad \bar{\partial}_b \hat{B} + [\hat{\mathcal{A}}^{0,1}, \hat{B}] = 0 \tag{6.29}$$

of hBF theory on $\mathcal{P}^2$. By repeating the discussion of section 5, one can show the equivalence of the data

$$(\mathcal{P}^{2|4}_M, \tilde{\varphi}_{+-}, \bar{\partial}_b) \;\sim\; (\mathcal{P}^{2|4}, \varphi_{+-} = \lambda_+ \mathbb{1}_4, \bar{\partial}_b + \hat{\mathcal{A}}^{0,1}) \;\sim\; (\mathcal{F}^{5|8}_M, \tilde{\varphi}_{+-}, \mathrm{d}_{\mathcal{T}_b})\,, \tag{6.30}$$

which extends the equivalences described in (6.25).

6.3 The deformed CR supertwistor space as a supermanifold

For developing a twistor correspondence involving the deformed CR supertwistor space $\mathcal{F}^{5|8}_M$, the description of $\mathcal{F}^{5|8}_M$ as a rank $0|4$ complex supervector bundle with a constant gauge potential (6.20) which twists the direct product of even and odd spaces is not sufficient. We rather have to interpret the total space of $\mathcal{F}^{5|8}_M$ as a supermanifold with deformed CR structure and deformed distribution $\mathcal{T}_M$.

Vector fields on $\mathcal{F}^{5|8}_M$. Remember that a covariant derivative along a vector field on the base space of a bundle can be lifted to a vector field on the total space of the bundle. In our case of the bundle (6.13), the lift of (6.18) reads as

$$\bar{W}^\pm_1 \tilde{\eta}^\pm_i = 0\,, \qquad \bar{W}^\pm_2 \tilde{\eta}^\pm_i = 0 \qquad \text{and} \qquad \left(\bar{W}^\pm_3 + M_k{}^j \tilde{\eta}^\pm_j \frac{\partial}{\partial \tilde{\eta}^\pm_k}\right) \tilde{\eta}^\pm_i = 0\,. \tag{6.31}$$

To see the explicit form of the vector fields corresponding to the integrable distribution

$$\mathcal{T}_M = \mathrm{span}\left\{ \frac{\partial}{\partial \bar{w}^a_\pm}, \frac{\partial}{\partial \bar{\tilde{\eta}}^\pm_i} \right\} \tag{6.32}$$

on $\mathcal{F}^{5|8}_M$, it is convenient to switch to the coordinates $(y^{\dot{\alpha}\dot{\beta}}, \lambda_\pm, \bar{\lambda}_\pm, \eta^{\dot{\alpha}}_i)$ by the formulæ

$$w^1_\pm = \lambda^\pm_{\dot{\alpha}} \lambda^\pm_{\dot{\beta}} y^{\dot{\alpha}\dot{\beta}}\,, \qquad w^2_\pm = \lambda_\pm \qquad \text{and} \qquad w^3_\pm = -\gamma_\pm \lambda^\pm_{\dot{\alpha}} \hat{\lambda}^\pm_{\dot{\beta}} y^{\dot{\alpha}\dot{\beta}}\,, \tag{6.33a}$$

$$\tilde{\eta}^+_i = \left(\mathrm{e}^{\lambda^+_{\dot{\alpha}} y^{\dot{\alpha}\dot{2}} M}\right)_i{}^j \eta^{\dot{\beta}}_j \lambda^+_{\dot{\beta}} \qquad \text{and} \qquad \tilde{\eta}^-_i = \left(\mathrm{e}^{-\lambda^-_{\dot{\alpha}} y^{\dot{\alpha}\dot{1}} M}\right)_i{}^j \eta^{\dot{\beta}}_j \lambda^-_{\dot{\beta}}\,. \tag{6.33b}$$

[26]These components can be derived from formula (6.36) given below.

By a straightforward calculation, we obtain

$$
\begin{aligned}
\mathrm{d}_{\mathcal{T}_M}|_{\tilde{\mathcal{V}}_\pm} &= \mathrm{d}\bar{w}^a_\pm \frac{\partial}{\partial \bar{w}^a_\pm} + \mathrm{d}\bar{\tilde{\eta}}^\pm_i \frac{\partial}{\partial \bar{\tilde{\eta}}^\pm_i} \\
&= \bar{\Theta}^1_\pm \bar{\mathcal{W}}^\pm_1 + \bar{\Theta}^2_\pm \bar{\mathcal{W}}^\pm_2 + (\bar{\Theta}^3_\pm \mp \gamma^2_\pm \hat{\lambda}^\pm_{\dot\alpha} \hat{\lambda}^\pm_{\dot\beta} y^{\dot\alpha\dot\beta} \Theta^2_\pm) \bar{\mathcal{W}}^\pm_3 + \bar{\mathcal{E}}^\pm_i \bar{V}^i_\pm \,,
\end{aligned}
\tag{6.34}
$$

where

$$
\begin{aligned}
\bar{\mathcal{W}}^\pm_1 &:= \bar{W}^\pm_1 \mp \lambda_\pm (T\bar{M}T)_i{}^j \hat{\lambda}^\pm_{\dot\alpha} \eta^{\dot\alpha}_j \bar{V}^i_\pm \,, \qquad \bar{\mathcal{W}}^\pm_2 \;:=\; \bar{W}^\pm_2 \,, \\
\bar{\mathcal{W}}^\pm_3 &:= \bar{W}^\pm_3 + \gamma_\pm (TM)_i{}^j \lambda^\pm_{\dot\alpha} \eta^{\dot\alpha}_j V^i_\pm + \gamma_\pm (T\bar{M}T)_i{}^j \hat{\lambda}^\pm_{\dot\alpha} \eta^{\dot\alpha}_j \bar{V}^i_\pm \,, \\
\bar{\mathcal{E}}^+_i &:= \bar{E}^+_i + \gamma_+ \hat{\lambda}^+_{\dot\alpha} \hat{\lambda}^+_{\dot\beta} \eta^{\dot\beta}_j (T\bar{M}T)_i{}^j \mathrm{d}y^{\dot1\dot\alpha} \,, \\
\bar{\mathcal{E}}^-_i &:= \bar{E}^-_i + \gamma_- \hat{\lambda}^-_{\dot\alpha} \hat{\lambda}^-_{\dot\beta} \eta^{\dot\beta}_j (T\bar{M}T)_i{}^j \mathrm{d}y^{\dot2\dot\alpha}
\end{aligned}
\tag{6.35}
$$

and $\bar{W}^\pm_a, V^i_\pm, \bar{V}^i_\pm$ and $\bar{\Theta}^a_\pm, \Theta^a_\pm$ were given in (3.34)–(3.36) and $(AB)_i{}^j := A_i{}^k B_k{}^j$. In fact, the formulæ (6.33b) and their inverses define a diffeomorphism between the supermanifolds $\mathcal{F}^{5|8}_M = (\mathbb{R}^{3|8} \times S^2, \mathcal{T}_M)$ and $\mathcal{F}^{5|8} = (\mathbb{R}^{3|8} \times S^2, \mathcal{T})$ which have different integrable distributions $\mathcal{T}_M$ and $\mathcal{T}$ (and different CR structures).

Vector fields on $\mathcal{P}^{2|4}_M$. In the above discussion, we used a transformation from the coordinates $\tilde{\eta}^\pm_i$ to the coordinates $\eta^\pm_i$ on $\mathcal{F}^{5|8}_M$, which are (pulled-back) sections of $\Pi\mathcal{O}(1)$. The corresponding splitting of the transition function was given in (6.14)–(6.16). One can find a similar splitting of the transition function (6.11) also on the complex supermanifold $\mathcal{P}^{2|4}_M$ and obtain new coordinates $\hat{\eta}^\pm_i$, which are sections of $\Pi\mathcal{O}(1)$, as well. Explicitly, we have

$$
\mathrm{e}^{\frac{w^1_+}{w^2_+}M} = \mathrm{e}^{\left(1-\frac{1}{1+w^2_+\bar{w}^2_+}\right)\frac{w^1_+}{w^2_+}M}\, \mathrm{e}^{\frac{w^1_+}{w^2_+(1+w^2_+\bar{w}^2_+)}M} = \mathrm{e}^{\frac{\bar{w}^2_+ w^1_+}{1+w^2_+\bar{w}^2_+}M}\, \mathrm{e}^{\frac{\bar{w}^2_- w^1_-}{1+w^2_-\bar{w}^2_-}M} \,,
\tag{6.36}
$$

which yields the formulæ

$$
\tilde{\eta}^+ = \mathrm{e}^{\frac{\bar{w}^2_+ w^1_+}{1+w^2_+\bar{w}^2_+}M}\, \hat{\eta}^+ \qquad \text{and} \qquad \tilde{\eta}^- = \mathrm{e}^{-\frac{\bar{w}^2_- w^1_-}{1+w^2_-\bar{w}^2_-}M}\, \hat{\eta}^- \,.
\tag{6.37}
$$

From this and (6.12), it follows that

$$
\hat{\eta}^+_i = w^2_+ \hat{\eta}^-_i
\tag{6.38}
$$

and these coordinates have the desired property. Furthermore, in the $(0,1)$ part of the differential

$$
\begin{aligned}
\bar{\partial}|_{\hat{\mathcal{V}}_\pm} &= \mathrm{d}\bar{w}^1_\pm \frac{\partial}{\partial \bar{w}^1_\pm} + \mathrm{d}\bar{w}^2_\pm \frac{\partial}{\partial \bar{w}^2_\pm} + \mathrm{d}\bar{\tilde{\eta}}^\pm_i \frac{\partial}{\partial \bar{\tilde{\eta}}^\pm_i} \qquad (6.39) \\
&= \mathrm{d}\bar{\hat{w}}^1_\pm \partial_{\bar{\hat{w}}^1_\pm} + \mathrm{d}\bar{\hat{w}}^2_\pm \left(\partial_{\bar{\hat{w}}^2_\pm} \mp \gamma^2_\pm \hat{w}^1_\pm M_i{}^j \hat{\eta}^\pm_j \partial_{\hat{\eta}^\pm_i}\right) + \left(\mathrm{d}\bar{\hat{\eta}}^\pm_i \pm \gamma^2_\pm \bar{\hat{w}}^1_\pm \bar{M}_i{}^j \bar{\hat{\eta}}^\pm_j \mathrm{d}\bar{\hat{w}}^2_\pm\right) \partial_{\bar{\hat{\eta}}^\pm_i} \,,
\end{aligned}
$$

where we introduced $\hat{w}^{1,2}_\pm = w^{1,2}_\pm$ for clarity, we see explicitly the deformation of the complex structure from $\mathcal{P}^{2|4}$ to $\mathcal{P}^{2|4}_M$. Note that the coordinates $\hat{\eta}^\pm_i$ can be pulled-back to $\mathcal{F}^{5|8}_M$ and are there related to the coordinates $\eta^\pm$ by

$$
\hat{\eta}^\pm = \mathrm{e}^{-w^3_\pm M} \eta^\pm \,.
\tag{6.40}
$$

6.4 Mass-deformed Bogomolny equations from phCS theory on $\mathcal{F}_M^{5|8}$

Holomorphic integral forms. The deformed mini-supertwistor space $\mathcal{P}_M^{2|4}$ fits into a double fibration

$$\begin{tikzcd} & \mathcal{F}_M^{5|8} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}_M^{2|4} & & \mathbb{R}^{3|8} \end{tikzcd} \tag{6.41}$$

similarly to the undeformed case $M = 0$. Recall that we had a holomorphic integral form on $\mathcal{P}^{2|4}$ locally defined by

$$\Omega|_{\hat{\mathcal{V}}_\pm} = \pm \mathrm{d}w_\pm^1 \wedge \mathrm{d}w_\pm^2 \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm \,. \tag{6.42}$$

One can extend Ω to a nonvanishing holomorphic volume form

$$\Omega^M|_{\hat{\mathcal{V}}_\pm} = \pm \mathrm{d}w_\pm^1 \wedge \mathrm{d}w_\pm^2 \mathrm{d}\tilde{\eta}_1^\pm \cdots \mathrm{d}\tilde{\eta}_4^\pm \tag{6.43}$$

on $\mathcal{P}_M^{2|4}$ if and only if $\operatorname{tr} M = 0$ [1]. This is the reason, why we imposed this condition from the very beginning. Similarly to the discussion of phCS theory on $\mathcal{F}^{5|8}$ in section 4, we consider a submanifold $\mathscr{X}_M \subset \mathcal{F}_M^{5|8}$ which is defined by the constraints $\bar{\tilde{\eta}}_i^\pm = 0$. Clearly, the latter equations are equivalent to $\bar{\eta}_i^\pm = 0$ and therefore $\mathscr{X}_M$ is diffeomorphic to $\mathscr{X}$. Note that the pull-back of the holomorphic integral form (6.43) to $\mathcal{F}_M^{5|8}$ coincides with $\pi_2^*\Omega$,

$$\tilde{\Omega}^M|_{\tilde{\mathcal{V}}_\pm} := \pi_2^*\Omega^M|_{\hat{\mathcal{V}}_\pm} = \pm\Theta_\pm^1 \wedge \Theta_\pm^2 \mathrm{d}\tilde{\eta}_1^\pm \cdots \mathrm{d}\tilde{\eta}_4^\pm = \pm\Theta_\pm^1 \wedge \Theta_\pm^2 \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm \tag{6.44}$$

which is due to (6.17) and the tracelessness of M.

Action functional. From here on, we proceed as in section 4 and consider a trivial rank n complex vector bundle over the CR supertwistor space $\mathcal{F}_M^{5|8}$ with a connection $\mathcal{A}_{\mathcal{T}_M}$ along the integrable distribution $\mathcal{T}_M$ defined in (6.32) and (6.34). By assuming that $\bar{V}_\pm^i \lrcorner \mathcal{A}_{\mathcal{T}_M} = 0$ and $\bar{V}_\pm^i(\bar{\mathcal{W}}_a^\pm \lrcorner \mathcal{A}_{\mathcal{T}_M}) = 0$, we may define the action functional

$$S_{\text{phCS}}^M = \int_{\mathscr{X}_M} \tilde{\Omega}^M \wedge \operatorname{tr}\left(\mathcal{A}_{\mathcal{T}_M} \wedge \mathrm{d}_{\mathcal{T}_M}\mathcal{A}_{\mathcal{T}_M} + \frac{2}{3}\mathcal{A}_{\mathcal{T}_M} \wedge \mathcal{A}_{\mathcal{T}_M} \wedge \mathcal{A}_{\mathcal{T}_M}\right) \tag{6.45}$$

of deformed phCS theory. The equations of motion keep the form (4.16) up to relabelling $\mathcal{T}$ by $\mathcal{T}_M$ and in components $\tilde{\mathcal{A}}_a^\pm := \bar{\mathcal{W}}_a^\pm \lrcorner \mathcal{A}_{\mathcal{T}_M}$, we have

$$\begin{aligned} \bar{W}_1^\pm \tilde{\mathcal{A}}_2^\pm - \bar{W}_2^\pm \tilde{\mathcal{A}}_1^\pm + [\tilde{\mathcal{A}}_1^\pm, \tilde{\mathcal{A}}_2^\pm] &= 0\,, \\ \bar{W}_2^\pm \tilde{\mathcal{A}}_3^\pm - \bar{W}_3^\pm \tilde{\mathcal{A}}_2^\pm + [\tilde{\mathcal{A}}_2^\pm, \tilde{\mathcal{A}}_3^\pm] \mp 2\gamma_\pm^2 \tilde{\mathcal{A}}_1^\pm &= M_j{}^i \eta_i^\pm \frac{\partial}{\partial \eta_j^\pm}\tilde{\mathcal{A}}_2^\pm\,, \\ \bar{W}_1^\pm \tilde{\mathcal{A}}_3^\pm - \bar{W}_3^\pm \tilde{\mathcal{A}}_1^\pm + [\tilde{\mathcal{A}}_1^\pm, \tilde{\mathcal{A}}_3^\pm] &= M_j{}^i \eta_i^\pm \frac{\partial}{\partial \eta_j^\pm}\tilde{\mathcal{A}}_1^\pm\,, \end{aligned} \tag{6.46}$$

where the vector fields (6.35) have already been substituted. The dependence of the components $\tilde{\mathcal{A}}_a^\pm$ on $\lambda_\pm, \bar{\lambda}_\pm$ and $\eta_i^\pm$ is of the same form as the one for $\mathcal{A}_a^\pm$ given in (4.20) but with coefficient functions obeying M-deformed equations. In the following, we will not put tildes over the coefficient functions for simplicity.

Mass-deformed super Bogomolny equations. Substituting the expansions of the form (4.20) for $\tilde{\mathcal{A}}_a^\pm$ and our vector fields $\bar{W}_a^\pm$ into (6.46), we obtain mass-deformed supersymmetric Bogomolny equations

$$
\begin{aligned}
f_{\dot\alpha\dot\beta} &= -\frac{\mathrm{i}}{2} D_{\dot\alpha\dot\beta}\Phi\,,\\
\varepsilon^{\dot\beta\dot\gamma} D_{\dot\alpha\dot\beta}\chi^i_{\dot\gamma} - \frac{1}{2} M_j{}^i \chi^j_{\dot\alpha} &= -\frac{\mathrm{i}}{2}[\Phi,\chi^i_{\dot\alpha}]\,,\\
\triangle\phi^{ij} + M_k{}^{[i} M_l{}^{j]}\phi^{kl} &= -\frac{1}{4}[\Phi,[\phi^{ij},\Phi]] - \mathrm{i} M_k{}^{[i}[\Phi,\phi^{j]k}] + \varepsilon^{\dot\alpha\dot\beta}\{\chi^i_{\dot\alpha},\chi^j_{\dot\beta}\}\,,\\
\varepsilon^{\dot\beta\dot\gamma} D_{\dot\alpha\dot\beta}\tilde\chi_{i\dot\gamma} - \frac{1}{2} M_i{}^j\tilde\chi_{j\dot\alpha} &= -\frac{\mathrm{i}}{2}[\tilde\chi_{i\dot\alpha},\Phi] + 2\mathrm{i}[\phi_{ij},\chi^j_{\dot\alpha}]\,,\\
\varepsilon^{\dot\beta\dot\gamma} D_{\dot\alpha\dot\beta} G_{\dot\gamma\dot\delta} &= -\frac{\mathrm{i}}{2}[G_{\dot\alpha\dot\delta},\Phi] + \mathrm{i}\{\chi^i_{\dot\alpha},\tilde\chi_{i\dot\delta}\} - \frac{1}{2}[\phi_{ij}, D_{\dot\alpha\dot\delta}\phi^{ij}]\ +\\
&\quad + \frac{\mathrm{i}}{4}\varepsilon_{\dot\alpha\dot\delta}[\phi_{ij},[\Phi,\phi^{ij}]] + \frac{1}{2}\varepsilon_{\dot\alpha\dot\delta} M_m{}^k[\phi_{kl},\phi^{lm}]\,,
\end{aligned}
\tag{6.47}
$$

where we have again abbreviated $\phi_{ij} := \frac{1}{2!}\varepsilon_{ijkl}\phi^{kl}$ and $\nabla_{\dot\alpha\dot\beta} = \partial_{(\dot\alpha\dot\beta)} + B_{\dot\alpha\dot\beta} = D_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi$.

The equations (6.46) show that, as in the undeformed case (4.17), some of the fields appearing in the expansions of $\tilde{\mathcal{A}}_a^\pm$ are not independent degrees of freedom but composite fields:

$$
\begin{aligned}
\phi^{ij}_{\dot\alpha\dot\beta} &= -\left(\partial_{(\dot\alpha\dot\beta)}\phi^{ij} + [B_{\dot\alpha\dot\beta},\phi^{ij}] - \varepsilon_{\dot\alpha\dot\beta} M_k{}^{[i}\phi^{j]k}\right),\\
\tilde\chi^{ijk}_{\dot\alpha(\dot\beta\dot\gamma)} &= -\frac{1}{2}\left(\partial_{(\dot\alpha(\dot\beta)}\tilde\chi^{ijk}_{\dot\gamma)} + [B_{\dot\alpha(\dot\beta},\tilde\chi^{ijk}_{\dot\gamma)}] + \frac{3}{2}\varepsilon_{\dot\alpha(\dot\beta} M_l{}^{[i}\chi^{jk]l}_{\dot\gamma)}\right),\\
G^{ijkl}_{\dot\alpha(\dot\beta\dot\gamma\dot\delta)} &= -\frac{1}{3}\left(\partial_{(\dot\alpha(\dot\beta)}G^{ijkl}_{\dot\gamma\dot\delta)} + [B_{\dot\alpha(\dot\beta},G^{ijkl}_{\dot\gamma\dot\delta)}] - 2\varepsilon_{\dot\alpha(\dot\beta} M_m{}^{[i}G^{jkl]m}_{\dot\gamma\dot\delta)}\right).
\end{aligned}
\tag{6.48}
$$

Finally, substituting our superfield expansions for $\tilde{\mathcal{A}}_a^\pm$ into the action (6.45) and integrating over the odd coordinates and over the Riemann sphere, we end up with

$$
S^M_{\mathrm{sB}} = S_{\mathrm{sB}} - \frac{1}{2}\int \mathrm{d}^3 y\ \mathrm{tr}\left\{\mathrm{i}\varepsilon^{\dot\alpha\dot\beta}\tilde\chi_{i\dot\alpha} M_j{}^i\chi^j_{\dot\beta} - \phi_{ij} M_k{}^i M_l{}^{[j}\phi^{k]l} + \mathrm{i}\Phi M_k{}^i[\phi_{ij},\phi^{jk}]\right\}\,, \tag{6.49}
$$

where $S_{\mathrm{sB}} = S^{M=0}_{\mathrm{sB}}$ is the action functional for the massless supersymmetric Bogomolny equations as given in (4.28).

To sum up, we have described a one-to-one correspondence between gauge equivalence classes of solutions to the supersymmetric Bogomolny equations with massive fermions and scalar fields and equivalence classes of $\mathcal{T}_M$-flat bundles over the CR supertwistor space $\mathcal{F}^{5|8}_M$ which are holomorphically trivial on each $\mathbb{C}P^1_{x,\eta}\hookrightarrow\mathcal{F}^{5|8}_M$. We have also described a one-to-one correspondence between the equivalence classes of $\mathcal{T}_M$-flat complex vector bundles over $\mathcal{F}^{5|8}_M$ and of holomorphic vector bundles over the deformed mini-supertwistor space $\mathcal{P}^{2|4}_M$. The assumption that these bundles become holomorphically trivial on projective lines translates in the Dolbeault description into a one-to-one correspondence between gauge equivalence classes of solutions to the field equations of *i*) hBF theory on the deformed mini-supertwistor space $\mathcal{P}^{2|4}_M$, *ii*) phCS theory on the CR supertwistor space $\mathcal{F}^{5|8}_M$ and *iii*) massive supersymmetric Bogomolny model on $\mathbb{R}^3$.

7. Examples of solutions to the super Bogomolny equations

In the preceding sections, we have presented in detail the relations between the supersymmetric Bogomolny equations on the euclidean space $\mathbb{R}^3$ and the field equations of phCS theory on the CR supertwistor space $\mathcal{F}^{5|8}$ as well as hBF theory on the mini-supertwistor space $\mathcal{P}^{2|4}$. We have shown that the moduli spaces of solutions to the field equations of these three theories are bijective. Furthermore, we introduced mass-deformed versions of these field theories. In this section, we want to show how the twistor correspondences described in the previous sections can be used for constructing explicit solutions to the supersymmetric Bogomolny equations. In fact, any solution to the standard Bogomolny equations given as a pair $(A_{\dot\alpha\dot\beta}, \Phi)$ of a gauge potential and a Higgs field can be extended to a solution including the remaining fields of the supersymmetrically extended Bogomolny equations in a nontrivial fashion. Here, we are not considering this task in full generality but just want to give some examples. For simplicity, we restrict ourselves to the case when only the fields $A_{\dot\alpha\dot\beta}, \Phi$ and $G_{\dot\alpha\dot\beta}$ are non-zero. In this case, the supersymmetric Bogomolny equations (4.25) simplify to

$$-\frac{1}{2}\varepsilon^{\dot\gamma\dot\delta}\left(\partial_{(\dot\alpha\dot\gamma)}A_{\dot\beta\dot\delta} - \partial_{(\dot\beta\dot\delta)}A_{\dot\alpha\dot\gamma} + [A_{\dot\alpha\dot\gamma}, A_{\dot\beta\dot\delta}]\right) = -\frac{\mathrm{i}}{2}\left(\partial_{(\dot\alpha\dot\beta)}\Phi + [A_{\dot\alpha\dot\beta}, \Phi]\right), \tag{7.1a}$$

$$\varepsilon^{\dot\gamma\dot\delta}\left(\partial_{(\dot\alpha\dot\gamma)}G_{\dot\beta\dot\delta} + [A_{\dot\alpha\dot\gamma}, G_{\dot\beta\dot\delta}]\right) = -\frac{\mathrm{i}}{2}[G_{\dot\alpha\dot\beta}, \Phi]\,. \tag{7.1b}$$

First, we discuss abelian solutions to these equations, which correspond to Dirac monopole-antimonopole systems. After this, we present two algorithms which generate non-abelian solutions.

7.1 Abelian solutions

Field equations. In the abelian case, (7.1) simplifies further to

$$\begin{aligned} \varepsilon^{\dot\gamma\dot\delta}\left(\partial_{(\dot\alpha\dot\gamma)}A_{\dot\beta\dot\delta} - \partial_{(\dot\beta\dot\delta)}A_{\dot\alpha\dot\gamma}\right) &= \mathrm{i}\partial_{(\dot\alpha\dot\beta)}\Phi\,, \\ \varepsilon^{\dot\gamma\dot\delta}\partial_{(\dot\alpha\dot\gamma)}G_{\dot\beta\dot\delta} &= 0\,. \end{aligned} \tag{7.2}$$

It is convenient to rewrite these equations in terms of the real coordinates x^r on $\mathbb{R}^3$ with $r = 1, 2, 3$ as

$$\frac{1}{2}\varepsilon_{rst}(\partial_s A_t - \partial_t A_s) = \partial_r\Phi\,, \tag{7.3a}$$

$$\partial_r G_r = 0\,, \tag{7.3b}$$

$$\varepsilon_{rst}\partial_s G_t = 0\,. \tag{7.3c}$$

From (7.3b), it follows that

$$G_r = \frac{1}{2}\varepsilon_{rst}(\partial_s\hat{A}_t - \partial_t\hat{A}_s)\,, \tag{7.4}$$

and from (7.3c), we obtain

$$G_r = -\partial_r\hat{\Phi}\,, \tag{7.5}$$

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where the sign in (7.5) was chosen to match the fact that in four dimensions, G_r corresponds to an anti-self-dual two-form with components $G_{\mu\nu} = \bar{\eta}^r_{\mu\nu} G_r$ and helicity -1, where $\bar{\eta}^r_{\mu\nu}$ are the 't Hooft tensors. Here, $\hat{A}_r$ and $\hat{\Phi}$ are a vector and a scalar, respectively. Therefore, the equations (7.3) can be rewritten as

$$\frac{1}{2}\varepsilon_{rst}(\partial_s A_t - \partial_t A_s) = \partial_r \Phi\,, \tag{7.6a}$$

$$\frac{1}{2}\varepsilon_{rst}(\partial_s \hat{A}_t - \partial_t \hat{A}_s) = -\partial_r \hat{\Phi}\,. \tag{7.6b}$$

It is well known that the equations (7.6a) describe Dirac monopoles while (7.6b) describe Dirac antimonopoles (see e.g. [33] and references therein). Thus, the action (4.28) with only the fields $f_{\dot{\alpha}\dot{\beta}}$, Φ and $G_{\dot{\alpha}\dot{\beta}}$ being non-zero can be considered as a proper action for the description of monopole-antimonopole systems.

Abelian monopole-antimonopole configurations. Let us consider a configuration of m_1 Dirac monopoles and m_2 antimonopoles located at points $a_i = (a_i^1, a_i^2, a_i^3)$ with $i = 1, \dots, m_1$ and $i = m_1{+}1, \dots, m_1{+}m_2$, respectively. Moreover, we assume for simplicity that $a_i^{1,2} \neq a_j^{1,2}$ for $i \neq j$. Such a configuration is then described by the fields

$$\begin{aligned} A^N &= \sum_{j=1}^{m_1} A^{N,j}\,, \qquad A^S = \sum_{j=1}^{m_1} A^{S,j}\,, \qquad \Phi^N = \Phi^S = \sum_{j=1}^{m_1} \frac{\mathrm{i}}{2r_j}\,, \\ \hat{A}^N &= \sum_{j=m_1+1}^{m_1+m_2} \bar{A}^{N,j}\,, \quad \hat{A}^S = \sum_{j=m_1+1}^{m_1+m_2} \bar{A}^{S,j}\,, \quad \hat{\Phi}^N = \hat{\Phi}^S = \sum_{j=m_1+1}^{m_1+m_2} \frac{\mathrm{i}}{2r_j}\,, \end{aligned} \tag{7.7}$$

where $A^{N,j} = A^{N,j}_m \mathrm{d}x^m$ and $A^{S,j} = A^{S,j}_m \mathrm{d}x^m$ with

$$\begin{aligned} A_1^{N,j} &= \frac{\mathrm{i}x_j^2}{2r_j(r_j + x_j^3)}\,, \quad A_2^{N,j} = \frac{-\mathrm{i}x_j^1}{2r_j(r_j + x_j^3)}\,, \quad A_3^{N,j} = 0\,, \\ A_1^{S,j} &= -\frac{\mathrm{i}x_j^2}{2r_j(r_j - x_j^3)}\,, \quad A_2^{S,j} = \frac{\mathrm{i}x_j^1}{2r_j(r_j - x_j^3)}\,, \quad A_3^{S,j} = 0\,, \end{aligned} \tag{7.8}$$

$$x_j^s = x^s - a_j^s\,, \qquad r_j^2 = \delta_{rs} x_j^r x_j^s\,. \tag{7.9}$$

Here, N and S denote the following two regions in $\mathbb{R}^3$:

$$\begin{aligned} \mathbb{R}^3_{N,m_1+m_2} &:= \mathbb{R}^3 \backslash \bigcup_{i=1}^{m_1+m_2} \{x^1 = a_i^1,\, x^2 = a_i^2,\, x^3 \le a_i^3\}\,, \\ \mathbb{R}^3_{S,m_1+m_2} &:= \mathbb{R}^3 \backslash \bigcup_{i=1}^{m_1+m_2} \{x^1 = a_i^1,\, x^2 = a_i^2,\, x^3 \ge a_i^3\} \end{aligned} \tag{7.10}$$

and the bar stands for complex conjugation. Note that

$$\mathbb{R}^3_{N,m_1+m_2} \cup \mathbb{R}^3_{S,m_1+m_2} = \mathbb{R}^3 \backslash \{a_1, \dots, a_{m_1+m_2}\} \tag{7.11}$$

and the configuration (7.7), (7.8) has delta-function sources at the points a_i with $i = 1, \dots, m_1 + m_2$.

7.2 Non-abelian solutions via a contour integral

For the gauge group SU(2), one can consider the Wu-Yang point monopole [37] and its generalizations to configurations describing m_1 monopoles and m_2 antimonopoles [38]. This solution, which is singular at points a_i, $i = 1, \ldots, m_1 + m_2$, is a solution to the equations (7.1) for su(2)-valued fields. However, it is just an abelian configuration in disguise, as it is equivalent to the multi-monopole configuration (7.7), (7.8) [38].

Solutions to linear equations. One can construct true non-abelian solutions to (7.1) as follows. Let us consider a configuration $A_{\dot\alpha\dot\beta} = 0 = \Phi$ and $G_{\dot\alpha\dot\beta} \neq 0$. Then from (7.1), one obtains the equation

$$\partial_{(\dot\alpha\dot\beta)} G^{\dot\beta\dot\gamma} = 0\,. \tag{7.12}$$

A large class of solutions to this equation can be described in the twistor approach [24] via the contour integral

$$G_0^{\dot\beta\dot\gamma} = \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}} \lambda_+^{\dot\beta}\lambda_+^{\dot\gamma}\Upsilon(\lambda_{\dot\alpha}^+\lambda_{\dot\delta}^+ y^{\dot\alpha\dot\delta}, \lambda_+)\,, \tag{7.13}$$

where $\Upsilon(w_+^1, w_+^2)$ is a Lie-algebra valued meromorphic function of $w_+^1 = \lambda_{\dot\alpha}^+\lambda_{\dot\beta}^+ y^{\dot\alpha\dot\beta}$ and $w_+^2 = \lambda_+$, holomorphic in the vicinity of the curve $\gamma \cong S^1 \subset \mathbb{C}P^1$. From (7.13), it follows that nontrivial contributions to $G_0^{\dot\beta\dot\gamma}$ are only given by those Υ which are elements of the cohomology group $H^1(\mathcal{P}^2, \mathcal{O}(-4) \otimes \mathrm{gl}(n,\mathbb{C}))$. It is easy to see that (7.13) satisfies (7.12) due to

$$\lambda_+^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\Upsilon = \frac{\partial\Upsilon}{\partial w_+^1}\lambda_+^{\dot\beta}\lambda_{\dot\beta}^+\lambda_{\dot\alpha}^+ = 0\,, \tag{7.14}$$

which appears after pulling the derivatives $\partial_{(\dot\alpha\dot\beta)}$ under the integral.

Dressed solutions. Consider now a fixed solution $(A_{\dot\alpha\dot\beta}, \Phi)$ of the Bogomolny equations (7.1a), e.g. the SU(2) BPS monopole [31, 30]. In the twistor approach, we can find functions $\hat\psi_\pm$ solving the linear system

$$\lambda_\pm^{\dot\beta}(\partial_{(\dot\alpha\dot\beta)} + A_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi)\hat\psi_\pm \;=\; 0 \qquad \text{and} \qquad \partial_{\bar\lambda_\pm}\hat\psi_\pm = 0\,, \tag{7.15}$$

which is equivalent to the linear system of phCS theory. These $\hat\psi_\pm$ are known explicitly for many cases, e.g. for our chosen example of the SU(2) BPS monopole [39]. Using $\hat\psi_\pm$, we can introduce "dressed" fields $G^{\dot\beta\dot\gamma}$ by the formula

$$G^{\dot\beta\dot\gamma} = \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}} \lambda_+^{\dot\beta}\lambda_+^{\dot\gamma}\hat\psi_+(y,\lambda_+)\Upsilon(y^{\dot\alpha\dot\delta}\lambda_{\dot\alpha}^+\lambda_{\dot\delta}^+, \lambda_+)\hat\psi_-^{-1}(y,\lambda_+^{-1})\,. \tag{7.16}$$

One can check that with this choice,

$$\partial_{(\dot\alpha\dot\beta)}G^{\dot\beta\dot\gamma} + \left[A_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi, G^{\dot\beta\dot\gamma}\right] \;=\; 0\,, \tag{7.17}$$

and therefore the configuration $(A_{\dot\alpha\dot\beta}, \Phi, G_{\dot\alpha\dot\beta})$ satisfies (7.1). The explicit form of a $G_{\dot\alpha\dot\beta}$ for a given $A_{\dot\alpha\dot\beta}$ and Φ is obtained from performing the contour integral (7.16) along γ after a proper choice of the Lie-algebra valued function Υ. Recall that the configuration (A_r, Φ)

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will be real, i.e., the fields will take values in the Lie algebra su(n), if the matrix-valued functions $\hat\psi_\pm$ in (7.15) satisfy the reality condition (2.51) and $\det(\hat\psi_+^{-1}\hat\psi_-)=1$. Imposing a proper reality condition on the function Υ will ensure the antihermiticity of G_r.

7.3 Solutions via nilpotent dressing transformations

In this section, we will present a novel algorithm for constructing solutions to the equations (7.1) based on the twistor description of hidden symmetry algebras in the supersymmetric SDYM theory in four dimensions [11].[27] Recall that we have described a one-to-one correspondence between equivalence classes $[\tilde f]$ of transition functions of $\mathcal{T}$-flat vector bundles $\tilde{\mathcal{E}}$ over the CR supertwistor space $\mathcal{F}^{5|8}$ obeying certain triviality conditions and gauge equivalence classes $[\hat{\mathcal{A}}]$ of solutions to the supersymmetric Bogomolny equations on $\mathbb{R}^3$. We can, however, associate with any open subset $\tilde{\mathcal{V}}_+\cap\tilde{\mathcal{V}}_-\subset\mathcal{F}^{5|8}$ an infinite number of such $[\tilde f]\in\mathcal{M}_{\rm phCS}$, which in turn yields an infinite number of $[\hat{\mathcal{A}}]\in\mathcal{M}_{\rm sB}$. Therefore, one naturally meets with a possibility of constructing new solutions from a given one (dressing transformation). In the following, we will discuss an example of such a construction but first we briefly recall the necessary background (for details, see e.g. [43, 44, 11]).

Linear system. We consider the linear system (4.40), which can be rewritten as[28]

$$(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm=0\,,\qquad \partial_{\bar\lambda_\pm}\hat\psi_\pm=0\qquad\text{and}\qquad(\bar V^i_\pm+\hat{\mathcal{A}}^i_\pm)\hat\psi_\pm=0\,,\tag{7.18}$$

where we have defined

$$\bar V^\pm_{\dot\alpha}:=\lambda^{\dot\beta}_\pm\partial_{(\dot\alpha\dot\beta)}\qquad\text{and}\qquad\hat{\mathcal{A}}^\pm_{\dot\alpha}:=\bar V^\pm_{\dot\alpha}\lrcorner\hat{\mathcal{A}}_{\mathcal{T}}\,.\tag{7.19}$$

From arguments similar to those used subsequent to (4.40), we have $\hat{\mathcal{A}}^\pm_{\dot\alpha}=\lambda^{\dot\beta}_\pm\mathcal{B}_{\dot\alpha\dot\beta}$ and $\hat{\mathcal{A}}^i_\pm=\lambda^{\dot\alpha}_\pm\mathcal{A}^i_{\dot\alpha}$ with λ-independent superfields $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$. The compatibility conditions for the linear system (7.18) are the equations (4.42). From this linear system, one also derives that $\tilde f_{+-}=\hat\psi_+^{-1}\hat\psi_-$ is $\mathcal{T}$-flat, i.e.

$$\bar V^+_{\dot\alpha}\tilde f_{+-}=0\,,\qquad\partial_{\bar\lambda_+}\tilde f_{+-}\;=\;0\qquad\text{and}\qquad\bar V^i_+\tilde f_{+-}=0\,.\tag{7.20}$$

Infinitesimal Riemann-Hilbert problem. The key idea is to study infinitesimal perturbations of the transition function $\tilde f_{+-}$ of the $\mathcal{T}$-flat vector bundle preserving (7.20) and the triviality properties discussed above. More explicitly, given such a function $\tilde f_{+-}=\hat\psi_+^{-1}\hat\psi_-$ (with $\partial_{\bar\lambda_\pm}\hat\psi_\pm=0$), we consider

$$\tilde f_{+-}+\delta\tilde f_{+-}=(\hat\psi_++\delta\hat\psi_+)^{-1}(\hat\psi_-+\delta\hat\psi_-)\,,\tag{7.21}$$

where δ represents some generic infinitesimal perturbation. Note that any infinitesimal $\mathcal{T}$-flat perturbation (i.e. preserving (7.20)) is allowed since for small perturbations, the trivializability property of the bundle $\tilde{\mathcal{E}}$ on the holomorphic curves $\mathbb{C}P^1_{x,\eta}\hookrightarrow\mathcal{F}^{5|8}$ is preserved (by a variant of Kodaira's theorem). Upon introducing the Lie-algebra valued function

$$\phi_{+-}\;:=\;\hat\psi_+(\delta\tilde f_{+-})\hat\psi_-^{-1}\tag{7.22}$$

[27] For an earlier account of hidden symmetry algebras in the context of SDYM theory, see e.g. [40–43].

[28] Note that (4.40a) and (4.40c) are equivalent to $\lambda^{\dot\alpha}_\pm(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm=0$ and $\hat\lambda^{\dot\alpha}_\pm(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm=0$, respectively, which together imply $(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm=0$.

and linearizing (7.21), we have to find a splitting

$$\phi_{+-} = \phi_+ - \phi_- \,, \tag{7.23}$$

where the Lie-algebra valued functions $\phi_\pm$ can be extended to holomorphic functions in $\lambda_\pm$, which yields

$$\delta\hat{\psi}_\pm = -\phi_\pm\hat{\psi}_\pm \,. \tag{7.24}$$

To find these $\phi_\pm$ from ϕ_{+-} means to solve the infinitesimal Riemann-Hilbert problem. Clearly, such solutions are not unique, as we have the freedom

$$\phi_{+-} = \phi_+ - \phi_- = (\phi_+ + \omega) - (\phi_- + \omega) \;=:\; \tilde{\phi}_+ - \tilde{\phi}_- \,, \tag{7.25}$$

with $\tilde{\phi}_\pm := \phi_\pm + \omega$, where the function ω does not depend on $\lambda_\pm$. This freedom can be used to preserve the transversal gauge condition (4.44), which is discussed in detail in appendix B.

Solutions to the linearized equations. Linearizing (7.18), we get

$$\delta\hat{\mathcal{A}}^+_{\dot\alpha} = \bar\nabla^+_{\dot\alpha}\phi_\pm \qquad \text{and} \qquad \delta\hat{\mathcal{A}}^i_+ = \bar\nabla^i_+\phi_\pm \,, \tag{7.26}$$

where we have introduced the operators $\bar\nabla^+_{\dot\alpha} := \bar V^+_{\dot\alpha} + \hat{\mathcal{A}}^+_{\dot\alpha}$ and $\bar\nabla^i_+ := \bar V^i_+ + \hat{\mathcal{A}}^i_+$. From (7.18), (7.20) and (7.21), it follows that

$$\bar\nabla^+_{\dot\alpha}\phi_{+-} = 0 = \bar\nabla^i_+\phi_{+-} \,, \tag{7.27}$$

and we eventually arrive at the formulæ

$$\delta\mathcal{B}_{\dot\alpha\dot\beta} = \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+\lambda_+^{\dot\beta}}\,\bar\nabla^+_{\dot\alpha}\phi_+ \qquad \text{and} \qquad \delta\mathcal{A}^i_{\dot\alpha} = \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+\lambda_+^{\dot\alpha}}\,\bar\nabla^i_+\phi_+ \,, \tag{7.28}$$

where the contour is $\gamma = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$. Thus, the consideration of infinitesimal perturbations of the transition function of some $\mathcal{T}$-flat vector bundle over the CR supertwistor space $\mathcal{F}^{5|8}$ obeying certain triviality conditions gives by virtue of the integral formulæ (7.28) infinitesimal deformations of the components $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$, which satisfy – by construction – the linearized supersymmetric Bogomolny equations (4.25). Once again, we have a one-to-one correspondence between equivalence classes of solutions, with equivalence induced on the gauge theory side by infinitesimal gauge transformations and on the twistor side by transformations of the form $\phi_\pm = \psi_\pm\chi_\pm\psi_\pm^{-1}$, where the $\chi_\pm$ are functions globally defined on $\tilde{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$ and annihilated by all vector fields from the distribution $\mathcal{T}$.

Nilpotent deformation of $\tilde{f}_{+-}$. Let us now exemplify our discussion by describing how to construct explicit solutions to (7.1). Consider a $\mathcal{T}$-flat vector bundle $\tilde{\mathcal{E}} \to \mathcal{F}^{5|8}$ of rank n which is holomorphically trivial when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{F}^{5|8}$. Assume further that a transition function $\tilde{f}_{+-}$ of $\tilde{\mathcal{E}}$ is chosen such that all the fields $\overset{\circ}{\chi}{}^i_{\dot\alpha}$, $\overset{\circ}{\phi}{}^{ij}$, $\overset{\circ}{\tilde\chi}_{i\dot\alpha}$ and $\overset{\circ}{G}_{\dot\alpha\dot\beta}$ vanish identically, i.e., we start with the field equation

$$\overset{\circ}{f}_{\dot\alpha\dot\beta} = -\frac{\mathrm{i}}{2}\overset{\circ}{D}_{\dot\alpha\dot\beta}\overset{\circ}{\Phi} \,. \tag{7.29}$$

Without loss of generality, we may assume that the transition function of $\tilde{\mathcal{E}}$ can be split as $\tilde{f}_{+-} = \hat{\psi}_+^{-1}\hat{\psi}_-$, where the $\hat{\psi}_\pm$ do not depend on the fermionic coordinates $\eta_i^\pm$.

Consider now the perturbation

$$\delta\tilde{f}_{+-} := -\frac{1}{4!}\varepsilon^{j_1\cdots j_4}\eta^+_{j_1}\cdots\eta^+_{j_4}[K,\tilde{f}_{+-}]\,, \tag{7.30}$$

where[29] $K \in \mathrm{gl}(n,\mathbb{C})$. Then a short calculation reveals that any splitting (7.23) is of the form

$$\phi_{+-} = \phi_+ - \phi_- = -\frac{1}{4!}\varepsilon^{j_1\cdots j_4}\eta^+_{j_1}\cdots\eta^+_{j_4}(\mathring{\phi}_+ - \mathring{\phi}_-)\,, \tag{7.31}$$

with $\mathring{\phi}_\pm := -[K,\hat{\psi}_\pm]\hat{\psi}_\pm^{-1}$. Introducing the shorthand notation $\eta^{\dot\gamma_1\cdots\dot\gamma_4} := -\frac{1}{4!}\varepsilon^{j_1\cdots j_4}\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_4}_{j_4}$, we find

$$\phi_{+-} = \eta^{\dot2\dot2\dot2\dot2}\mathring{\phi}^4_{+-} + 4\eta^{\dot2\dot2\dot2\dot1}\mathring{\phi}^3_{+-} + 6\eta^{\dot2\dot2\dot1\dot1}\mathring{\phi}^2_{+-} + 4\eta^{\dot2\dot1\dot1\dot1}\mathring{\phi}^1_{+-} + \eta^{\dot1\dot1\dot1\dot1}\mathring{\phi}^0_{+-}\,, \tag{7.32}$$

where we have used the fact that $\eta^{\dot\gamma_1\cdots\dot\gamma_4}$ is totally symmetric and defined

$$\mathring{\phi}^m_{+-} \;:=\; \lambda^m_+\mathring{\phi}_+ - \lambda^m_+\mathring{\phi}_- \;:=\; \mathring{\phi}^m_+ - \mathring{\phi}^m_-\,. \tag{7.33}$$

The functions $\mathring{\phi}^m_\pm$ can be expanded as $(m \geq 0)$

$$\mathring{\phi}^m_\pm = \sum_{n=0}^{\infty}\lambda_+^{\pm n}\mathring{\phi}^{m(n)}_\pm \tag{7.34}$$

with

$$\mathring{\phi}^{m(n)}_+ = \begin{cases}\delta_{m,0}\mathring{\phi}^{0(0)}_+ & n=0\\ \mathring{\phi}^{0(n-m)}_+ - \mathring{\phi}^{0(m-n)}_- & n>0\end{cases} \qquad \text{and} \qquad \mathring{\phi}^{m(n)}_- = \mathring{\phi}^{0(m+n)}_-\,. \tag{7.35}$$

Combining the expansion

$$\phi_\pm = \sum_{n=0}^{\infty}\lambda_+^{\pm n}\phi^{(n)}_\pm \tag{7.36}$$

with (7.31)–(7.35), we therefore find

$$\phi^{(n)}_- = \eta^{\dot2\dot2\dot2\dot2}\mathring{\phi}^{0(4+n)}_- + 4\eta^{\dot2\dot2\dot2\dot1}\mathring{\phi}^{0(3+n)}_- + 6\eta^{\dot2\dot2\dot1\dot1}\mathring{\phi}^{0(2+n)}_- + 4\eta^{\dot2\dot1\dot1\dot1}\mathring{\phi}^{0(1+n)}_- + \eta^{\dot1\dot1\dot1\dot1}\mathring{\phi}^{0(n)}_-\,, \tag{7.37}$$

and a similar expression for $\phi^{(n)}_+$. At this point, we have to choose an ω which guarantees that the transversal gauge condition is satisfied. Explicitly, a possible ω is given by

$$\omega = -\eta^{\dot2\dot2\dot2\dot1}\mathring{\phi}^{0(3)}_- - 3\eta^{\dot2\dot2\dot1\dot1}\mathring{\phi}^{0(2)}_- - 3\eta^{\dot2\dot1\dot1\dot1}\mathring{\phi}^{0(1)}_- - \eta^{\dot1\dot1\dot1\dot1}\mathring{\phi}^{0(0)}_-\,, \tag{7.38}$$

which is derived in appendix B, where also a detailed discussion of this point is found.

[29] Considering vector bundles subject to the reality conditions induced by (2.51), one restricts the perturbations to those preserving these conditions. In our subsequent example, they read explicitly

$$\delta\tilde{f}_{+-} = -\frac{1}{4!}\varepsilon^{j_1\cdots j_4}\left(\eta^+_{j_1}\cdots\eta^+_{j_4} + \eta^-_{j_1}\cdots\eta^-_{j_4}\right)[K,\tilde{f}_{+-}]$$

with $K \in \mathrm{su}(n)$.

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Towards explicit solutions. The infinitesimal perturbations $\delta\mathcal{B}_{\dot\alpha\dot\beta}$ and $\delta\mathcal{A}^i_{\dot\alpha}$ are obtained upon integration of the formulæ (7.28):

$$\begin{aligned}\delta\mathcal{B}_{\dot\alpha\dot 1} &= \overset{\circ}{\nabla}_{\dot\alpha\dot 1}(\phi_-^{(0)}+\omega) = \overset{\circ}{\nabla}_{\dot\alpha\dot 1}(\phi_+^{(0)}+\omega) - \overset{\circ}{\nabla}_{\dot\alpha\dot 2}\phi_+^{(1)}\,,\\ \delta\mathcal{B}_{\dot\alpha\dot 2} &= -\overset{\circ}{\nabla}_{\dot\alpha\dot 1}\phi_-^{(1)} + \overset{\circ}{\nabla}_{\dot\alpha\dot 2}(\phi_-^{(0)}+\omega) = \overset{\circ}{\nabla}_{\dot\alpha\dot 2}(\phi_+^{(0)}+\omega)\end{aligned} \tag{7.39a}$$

and

$$\begin{aligned}\delta\mathcal{A}^i_{\dot 1} &= \partial^i_{\dot 1}(\phi_-^{(0)}+\omega) = \partial^i_{\dot 1}(\phi_+^{(0)}+\omega) - \partial^i_{\dot 2}\phi_+^{(1)}\,,\\ \delta\mathcal{A}^i_{\dot 2} &= -\partial^i_{\dot 1}\phi_-^{(1)} + \partial^i_{\dot 2}(\phi_-^{(0)}+\omega) = \partial^i_{\dot 2}(\phi_+^{(0)}+\omega)\,.\end{aligned} \tag{7.39b}$$

Consider now the expansions

$$\begin{aligned}\delta\mathcal{B}_{\dot\alpha\dot\beta} &= \delta\overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta} + \sum_{k\geq 1}\frac{1}{k!}\,\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_k}_{j_k}\,\delta[\overset{\circ}{\dot\alpha\dot\beta}]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}\,,\\ \delta\mathcal{A}^i_{\dot\alpha} &= \sum_{k\geq 1}\frac{k}{(k+1)!}\,\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_k}_{j_k}\,\delta[\overset{\circ}{\underset{\dot\alpha}{i}}]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}\,,\end{aligned} \tag{7.40}$$

where the brackets $[\ \ ]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}$ are composite expressions of some superfields, cf. also appendix A. Since our particular deformation of the transition function implies that $\phi_\pm+\omega = \mathcal{O}(\eta^4)$, the resulting deformations of $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$ are of the form $\mathcal{B}_{\dot\alpha\dot\beta} = \mathcal{O}(\eta^4)$ and $\mathcal{A}^i_{\dot\alpha} = \mathcal{O}(\eta^3)$, respectively. In transversal gauge, the explicit superfield expansions (4.47) show that $\delta\overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta} = \delta\overset{\circ}{\chi}{}^i_{\dot\alpha} = \delta\overset{\circ}{\phi}{}^{ij} = \delta\overset{\circ}{\tilde\chi}_{i\dot\alpha} = 0$. Together with the recursion relations (4.46), they moreover imply that the variation of all higher order terms (than those given in (4.47)) in the η-expansions vanish. Hence, from (7.40) we find

$$\begin{aligned}\delta\mathcal{B}_{\dot\alpha\dot\beta} &= \frac{1}{2\cdot 4!}\varepsilon^{j_1j_2j_3j_4}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\eta^{\dot\gamma_4}_{j_4}\varepsilon_{\dot\beta\dot\gamma_1}\overset{\circ}{\nabla}_{\dot\alpha\dot\gamma_2}\delta\overset{\circ}{G}_{\dot\gamma_3\dot\gamma_4}\,,\\ \delta\mathcal{A}^i_{\dot\alpha} &= \frac{3}{4!}\varepsilon^{ij_1j_2j_3}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\varepsilon_{\dot\alpha\dot\gamma_1}\delta\overset{\circ}{G}_{\dot\gamma_2\dot\gamma_3}\,.\end{aligned} \tag{7.41}$$

Comparing these equations with (7.39) and the η-expansions of $\phi_\pm^{(0)}$, $\phi_\pm^{(1)}$ and ω given earlier, we arrive at

$$\delta\overset{\circ}{G}_{\dot 1\dot 1} = 2\overset{\circ}{\phi}{}^{0(1)}_-\,,\qquad \delta\overset{\circ}{G}_{\dot 1\dot 2} = 2\overset{\circ}{\phi}{}^{0(2)}_-\qquad\text{and}\qquad \delta\overset{\circ}{G}_{\dot 2\dot 2} = 2\overset{\circ}{\phi}{}^{0(3)}_- \tag{7.42}$$

together with the field equations

$$\overset{\circ}{f}_{\dot\alpha\dot\beta} = -\frac{\mathrm{i}}{2}\overset{\circ}{D}_{\dot\alpha\dot\beta}\overset{\circ}{\Phi}\qquad\text{and}\qquad \varepsilon^{\dot\beta\dot\gamma}\overset{\circ}{D}_{\dot\alpha\dot\beta}\delta\overset{\circ}{G}_{\dot\gamma\dot\delta} \;=\; -\frac{\mathrm{i}}{2}[\delta\overset{\circ}{G}_{\dot\alpha\dot\delta},\overset{\circ}{\Phi}]\,. \tag{7.43}$$

Since the equations (7.1) are linear in $G_{\dot\alpha\dot\beta}$, we hence have generated a solution

$$(A_{\dot\alpha\dot\beta} := \overset{\circ}{A}_{\dot\alpha\dot\beta}, \Phi := \overset{\circ}{\Phi}, G_{\dot\alpha\dot\beta} := \delta\overset{\circ}{G}_{\dot\alpha\dot\beta}) \tag{7.44}$$

to (7.1) starting from a solution to (7.1a). Thus, knowing the explicit splitting $\tilde f_{+-} = \hat\psi_+^{-1}\hat\psi_-$, we can define functions $\overset{\circ}{\phi}_\pm = -[K,\hat\psi_\pm]\hat\psi_\pm^{-1}$ which then in turn yield $G_{\dot\alpha\dot\beta}$.

8. Comments on $\mathcal{N}=8$ SYM theory in three dimensions

The full $\mathcal{N}=8$ SYM theory in three dimensions is slightly out of the scope of this paper, but as an outlook, we would like to sketch the construction of a supertwistor correspondence for this theory and leave the details to future work.

Twistor description of $\mathcal{N}=3$ SYM theory. Recall that there is a one-to-one correspondence between gauge equivalence classes of solutions to the $\mathcal{N}=3$ SYM equations in four complex dimensions and equivalence classes of holomorphic vector bundles $\mathcal{E}$ over a quadric $\mathcal{L}^{5|6}$ in (an open subset of) the product space $\mathbb{C}P^{3|3}\times\mathbb{C}P^{3|3}_*$ [45] such that the bundles $\mathcal{E}$ are holomorphically trivial on each submanifold $\mathcal{L}^{2|0}_{x,\theta,\eta}\cong\mathbb{C}P^1\times\mathbb{C}P^1_*\hookrightarrow\mathcal{L}^{5|6}$ with $(x,\eta,\theta)\in\mathbb{C}^{4|12}$. The space $\mathcal{L}^{5|6}$ is also known as the super-ambitwistor space[30] and enters into the double fibration

$$\begin{tikzcd} & \mathcal{F}^{6|12} \arrow[dl] \arrow[dr] & \\ \mathcal{L}^{5|6} & & \mathbb{C}^{4|12} \end{tikzcd} \tag{8.1}$$

where $\mathcal{F}^{6|12}\cong\mathbb{C}^{4|12}\times\mathbb{C}P^1\times\mathbb{C}P^1_*$ is the correspondence space with coordinates $(x^{\alpha\dot\alpha},\theta^{\alpha i},\eta_i^{\dot\alpha})$ on $\mathbb{C}^{4|12}$ and homogeneous coordinates $\lambda_{\dot\alpha}$ on $\mathbb{C}P^1$ and μ_α on $\mathbb{C}P^1_*$, respectively. The tangent spaces to the $(1|6)$-dimensional leaves of the fibration $\mathcal{F}^{6|12}\to\mathcal{L}^{5|6}$ are spanned by the holomorphic vector fields

$$\mu^\alpha\lambda^{\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}}\,,\qquad \lambda^{\dot\alpha}\left(\frac{\partial}{\partial\eta_i^{\dot\alpha}}+\theta^{\alpha i}\frac{\partial}{\partial x^{\alpha\dot\alpha}}\right)\qquad\text{and}\qquad \mu^\alpha\left(\frac{\partial}{\partial\theta^{\alpha i}}+\eta_i^{\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}}\right) \tag{8.2}$$

which enter into the linear system

$$\begin{gathered}\mu^\alpha\lambda^{\dot\alpha}(\partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha\dot\alpha})\psi=0\,,\\ \lambda^{\dot\alpha}\left(\frac{\partial}{\partial\eta_i^{\dot\alpha}}+\theta^{\alpha i}\partial_{\alpha\dot\alpha}+\mathcal{A}^i_{\dot\alpha}\right)\psi=0\,,\qquad \mu^\alpha\left(\frac{\partial}{\partial\theta^{\alpha i}}+\eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha i}\right)\psi=0\,,\end{gathered} \tag{8.3}$$

where ψ is a locally defined $\mathrm{GL}(n,\mathbb{C})$-valued function on $\mathcal{F}^{6|12}$. The compatibility conditions of (8.3) are equivalent to the $\mathcal{N}=3$ SYM field equations. On the other hand, these equations are also equivalent to the hCS field equations on $\mathcal{L}^{5|6}$ whose solutions describe holomorphic structures on the complex vector bundle $\mathcal{E}\to\mathcal{L}^{5|6}$ [2, 3].

Reduced twistor correspondence. Similarly to the supersymmetric Bogomolny model which was obtained by a dimensional reduction of $\mathcal{N}=4$ SDYM theory, one can establish a twistor correspondence for the full $\mathcal{N}=8$ SYM theory in three (complex) dimensions by using a dimensional reduction of $\mathcal{L}^{5|6}$. Taking the quotient of the spaces in the diagram (8.1) with respect to the actions of the abelian groups generated by the vector field $\mathscr{T}_4$, we

[30] See also [46] for the bosonic case. For a recent review and further references, see [3].

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arrive at the diagram

$$
\begin{tikzcd}
& \mathcal{F}^{6|12} \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathcal{L}^{5|6} \arrow[dd] & & \mathbb{C}^{4|12} \arrow[dd] \\
& \mathcal{F}^{5|12} \arrow[dl] \arrow[dr] & \\
\mathcal{L}^{4|6} & & \mathbb{C}^{3|12}
\end{tikzcd}
\tag{8.4}
$$

describing the reduction of the supertwistor correspondence (8.1). Recall that in three dimensions, we have the vector fields (3.4) and the decomposition (3.5). Substituting the latter into (8.2) and assuming independence of all functions of x^4, we obtain the vector fields

$$
\mu^{\dot\alpha}\lambda^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\,, \qquad \lambda^{\dot\beta}\left(\frac{\partial}{\partial\eta_i^{\dot\beta}} + \theta^{\dot\alpha i}\partial_{(\dot\alpha\dot\beta)}\right) \qquad \text{and} \qquad \mu^{\dot\alpha}\left(\frac{\partial}{\partial\theta^{\dot\alpha i}} + \eta_i^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\right)\,, \tag{8.5}
$$

which are tangent to the $(1|6)$-dimensional leaves of the fibration $\mathcal{F}^{5|12} \to \mathcal{L}^{4|6}$, where $\mathcal{F}^{5|12} \cong \mathbb{C}^{3|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$.

Constraint equations. Similarly, the linear system (8.3) is reduced to

$$
\begin{gathered}
\mu^{\dot\alpha}\lambda^{\dot\beta}(\partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta})\psi = 0\,,\\
\lambda^{\dot\beta}\left(\frac{\partial}{\partial\eta_i^{\dot\beta}} + \theta^{\dot\alpha i}\partial_{(\dot\alpha\dot\beta)} + \mathcal{A}^i_{\dot\beta}\right)\psi = 0\,, \qquad \mu^{\dot\alpha}\left(\frac{\partial}{\partial\theta^{\dot\alpha i}} + \eta_i^{\dot\beta}\partial_{(\dot\alpha\dot\beta)} + \mathcal{A}_{\dot\alpha i}\right)\psi = 0\,.
\end{gathered}
\tag{8.6}
$$

The corresponding compatibility conditions read as

$$
\begin{gathered}
\{\nabla_{\dot\alpha i}, \nabla^j_{\dot\beta}\} - 2\delta_i^j\nabla_{\dot\alpha\dot\beta} = 0\,,\\
\{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} + \{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\} \ = 0\,, \qquad \{\nabla_{\dot\alpha i}, \nabla_{\dot\beta j}\} + \{\nabla_{\dot\beta i}, \nabla_{\dot\alpha j}\} = 0\,,
\end{gathered}
\tag{8.7}
$$

where we introduced the differential operators

$$
\begin{gathered}
\nabla_{\dot\alpha\dot\beta} := \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}\,,\\
\nabla^i_{\dot\beta} := \frac{\partial}{\partial\eta_i^{\dot\beta}} + \theta^{\dot\alpha i}\partial_{(\dot\alpha\dot\beta)} + \mathcal{A}^i_{\dot\beta}\,, \qquad \nabla_{\dot\alpha i} := \frac{\partial}{\partial\theta^{\dot\alpha i}} + \eta_i^{\dot\beta}\partial_{(\dot\alpha\dot\beta)} + \mathcal{A}_{\dot\alpha i}\,.
\end{gathered}
\tag{8.8}
$$

The equations (8.7) are the constraint equations of $\mathcal{N} = 3$ SYM theory in four dimensions reduced to three dimensions.

Outlook. It remains to clarify the geometry of the mini-superambitwistor space $\mathcal{L}^{4|6}$ together with its real structure τ. In case that $\mathcal{L}^{4|6}$ is a Calabi-Yau supermanifold (for which there are certain hints), one can define an open topological B-model on this space, which is supposed to describe holomorphic structures on complex vector bundles over $\mathcal{L}^{4|6}$. These can then be linked to solutions of $\mathcal{N} = 3$ SYM theory reduced to three dimensions. The latter theory should (under an additional assumption) be equivalent to $\mathcal{N} = 8$ SYM theory similarly to the equivalence of $\mathcal{N} = 3$ and $\mathcal{N} = 4$ SYM theories in four dimensions. Note that on $\mathcal{L}^{4|6}$, one cannot impose the euclidean reality condition (2.31) on the fermionic coordinates. Only two other conditions leading to kleinian and Minkowski signature, respectively, are consistent [3].

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A. Superfield expansions

Field equations. In section 4.4, we obtained the constraint equations (4.43) for the differential operators $\nabla_{\dot\alpha\dot\beta} := \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}$ and $D^i_{\dot\alpha} = \frac{\partial}{\partial\eta^{\dot\alpha}_i} + \mathcal{A}^i_{\dot\alpha}$ which are equivalent to the supersymmetric Bogomolny equations

$$\begin{aligned}
f_{\dot\alpha\dot\beta} &= -\frac{\mathrm{i}}{2} D_{\dot\alpha\dot\beta}\Phi\,,\\
\varepsilon^{\dot\beta\dot\gamma} D_{\dot\alpha\dot\beta}\chi^i_{\dot\gamma} &= -\frac{\mathrm{i}}{2}[\Phi,\chi^i_{\dot\alpha}]\,,\\
\triangle\phi^{ij} &= -\frac{1}{4}[\Phi,[\phi^{ij},\Phi]] + \varepsilon^{\dot\alpha\dot\beta}\{\chi^i_{\dot\alpha},\chi^j_{\dot\beta}\}\,,\\
\varepsilon^{\dot\beta\dot\gamma} D_{\dot\alpha\dot\beta}\tilde\chi_{i\dot\gamma} &= -\frac{\mathrm{i}}{2}[\tilde\chi_{i\dot\alpha},\Phi] + 2\mathrm{i}[\phi_{ij},\chi^j_{\dot\alpha}]\,,\\
\varepsilon^{\dot\beta\dot\gamma} D_{\dot\alpha\dot\beta} G_{\dot\gamma\dot\delta} &= -\frac{\mathrm{i}}{2}[G_{\dot\alpha\dot\delta},\Phi] + \mathrm{i}\{\chi^i_{\dot\alpha},\tilde\chi_{i\dot\delta}\} - \frac{1}{2}[\phi_{ij}, D_{\dot\alpha\dot\delta}\phi^{ij}] + \frac{\mathrm{i}}{4}\varepsilon_{\dot\alpha\dot\delta}[\phi_{ij},[\Phi,\phi^{ij}]]
\end{aligned}\tag{A.1}$$

with all the fields being superfields and defined by

$$\begin{aligned}
\mathcal{A}_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi &:= \mathcal{B}_{\dot\alpha\dot\beta}\,,\\
\chi^i_{\dot\alpha} &:= \Sigma^i_{\dot\alpha}\,,\\
\phi^{ij} &:= \Sigma^{ij}\,,\\
\tilde\chi_{i\dot\alpha} &:= \frac{1}{3}\varepsilon_{ijkl} D^j_{\dot\alpha}\phi^{kl}\,,\\
G_{\dot\alpha\dot\beta} &:= -\frac{1}{4} D^i_{(\dot\alpha}\tilde\chi_{i\dot\beta)}\,.
\end{aligned}\tag{A.2}$$

Superfield expansions. To prove that the equations (A.1) are equivalent to the supersymmetric Bogomolny equations (4.25), we need the explicit superfield expansions of $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$ from which all remaining expansions can be derived by using the constraint equations (4.43). Since the proofs of the subsequent general assertions are rather straightforward, we leave them to the interested reader.

Consider some generic superfield φ. Its explicit η-expansion looks as

$$\varphi = \mathring{\varphi} + \sum_{k\geq 1}\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_k}_{j_k}\,\varphi^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}\,.\tag{A.3}$$

Furthermore, we have $\mathcal{D}\varphi = \eta^{\dot\gamma_1}_{j_1}[\ \ ]^{j_1}_{\dot\gamma_1}$, where the bracket $[\ \ ]^{j_1}_{\dot\gamma_1}$ is a composite expression of some superfields.[31] Let

$$\mathcal{D}[\ \ ]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k} = \eta^{\dot\gamma_{k+1}}_{j_{k+1}}[\ \ ]^{j_1\cdots j_{k+1}}_{\dot\gamma_1\cdots\dot\gamma_{k+1}}\,.\tag{A.4}$$

[31] For example, $\mathcal{D}\mathcal{B}_{\dot\alpha\dot\beta} = \eta^{\dot\gamma_1}_{j_1}[\dot\alpha\dot\beta]^{j_1}_{\dot\gamma_1}$, with $[\dot\alpha\dot\beta]^{j_1}_{\dot\gamma_1} = -\varepsilon_{\dot\beta\dot\gamma_1}\chi^{j_1}_{\dot\alpha}$.

Then we have

$$\varphi = \mathring{\varphi} + \sum_{k\geq 1} \frac{1}{k!}\, \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, \overset{\circ}{[\ \ ]}{}^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}\,, \tag{A.5}$$

which follows by induction. In case the recursion relation of φ was given by $(1+\mathcal{D})\varphi = \eta^{\dot\gamma_1}_{j_1}[\ \ ]^{j_1}_{\dot\gamma_1}$, as it happens to be for $\mathcal{A}^i_{\dot\alpha}$, then $\mathring{\varphi} = 0$ and the superfield expansion is of the form

$$\varphi = \sum_{k\geq 1} \frac{k}{(k+1)!}\, \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, \overset{\circ}{[\ \ ]}{}^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}\,. \tag{A.6}$$

From this, it is a bit tedious but straightforward to determine the superfield expansions of $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$. For

$$\mathcal{B}_{\dot\alpha\dot\beta} = \mathring{\mathcal{B}}_{\dot\alpha\dot\beta} + \sum_{k\geq 1} \frac{1}{k!}\, \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, \overset{\circ}{[\dot\alpha\dot\beta]}{}^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}\,, \tag{A.7}$$

we obtain as the first four coefficients

$$\begin{aligned}
[\dot\alpha\dot\beta]^{j_1}_{\dot\gamma_1} &= -\mathrm{i}\varepsilon_{\dot\beta\dot\gamma_1}\chi^{j_1}_{\dot\alpha}\,,\\
[\dot\alpha\dot\beta]^{j_1j_2}_{\dot\gamma_1\dot\gamma_2} &= \varepsilon_{\dot\beta\dot\gamma_1}\nabla_{\dot\alpha\dot\gamma_2}\phi^{j_1j_2}\,,\\
[\dot\alpha\dot\beta]^{j_1j_2j_3}_{\dot\gamma_1\dot\gamma_2\dot\gamma_3} &= -\frac{1}{2}\varepsilon_{\dot\beta\dot\gamma_1}\varepsilon^{j_1j_2j_3k}\nabla_{\dot\alpha\dot\gamma_2}\tilde\chi_{k\dot\gamma_3} - \mathrm{i}\varepsilon_{\dot\beta\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_3}[\chi^{j_3}_{\dot\alpha},\phi^{j_1j_2}]\,,\\
[\dot\alpha\dot\beta]^{j_1j_2j_3j_4}_{\dot\gamma_1\dot\gamma_2\dot\gamma_3\dot\gamma_4} &= -\frac{1}{2}\varepsilon_{\dot\beta\dot\gamma_1}\varepsilon^{j_1j_2j_3j_4}\nabla_{\dot\alpha\dot\gamma_2}G_{\dot\gamma_3\dot\gamma_4} + \frac{1}{2}\varepsilon_{\dot\beta\dot\gamma_1}\varepsilon_{\dot\gamma_3\dot\gamma_4}\varepsilon^{j_1j_2j_3k}\nabla_{\dot\alpha\dot\gamma_2}[\phi^{j_4l},\phi_{kl}]\ +\\
&\quad + \frac{\mathrm{i}}{2}\varepsilon_{\dot\beta\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_4}\varepsilon^{j_1j_2j_3k}\{\chi^{j_4}_{\dot\alpha},\tilde\chi_{k\dot\gamma_3}\} + \varepsilon_{\dot\beta\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_3}[\nabla_{\dot\alpha\dot\gamma_4}\phi^{j_3j_4},\phi^{j_1j_2}]\ -\\
&\quad - \frac{\mathrm{i}}{2}\varepsilon_{\dot\beta\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_3}\varepsilon^{j_1j_2j_3k}\{\chi^{j_3}_{\dot\alpha},\tilde\chi_{k\dot\gamma_4}\}\,,
\end{aligned} \tag{A.8}$$

where we have heavily used the recursion relations (4.46). Quite similarly, we find for

$$\mathcal{A}^i_{\dot\alpha} = \sum_{k\geq 1} \frac{k}{(k+1)!}\, \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, \overset{\circ}{[{}^{\,i}_{\dot\alpha}]}{}^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k} \tag{A.9}$$

the following coefficients:

$$[{}^{\,i}_{\dot\alpha}]^{j_1}_{\dot\gamma_1} = \varepsilon_{\dot\alpha\dot\gamma_1}\phi^{ij_1}\,, \tag{A.10}$$

$$\begin{aligned}
[{}^{\,i}_{\dot\alpha}]^{j_1j_2}_{\dot\gamma_1\dot\gamma_2} &= -\frac{1}{2}\varepsilon_{\dot\alpha\dot\gamma_1}\varepsilon^{ij_1j_2k}\tilde\chi_{k\dot\gamma_2}\,,\\
[{}^{\,i}_{\dot\alpha}]^{j_1j_2j_3}_{\dot\gamma_1\dot\gamma_2\dot\gamma_3} &= \frac{1}{2}\varepsilon_{\dot\alpha\dot\gamma_1}\varepsilon^{ij_1j_2j_3}G_{\dot\gamma_2\dot\gamma_3} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_3}\varepsilon^{ij_1j_2k}[\phi^{j_3l},\phi_{kl}]\,,\\
[{}^{\,i}_{\dot\alpha}]^{j_1j_2j_3j_4}_{\dot\gamma_1\dot\gamma_2\dot\gamma_3\dot\gamma_4} &= \frac{1}{2}\varepsilon_{\dot\alpha\dot\gamma_1}\varepsilon_{\dot\gamma_4(\dot\gamma_2}\varepsilon^{ij_1j_2j_3}[\tilde\chi_{k\dot\gamma_3)},\phi^{j_4k}] - \frac{1}{4}\varepsilon_{\dot\alpha\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_3}\varepsilon^{ij_1j_2k}\varepsilon^{j_3lj_4m}[\tilde\chi_{m\dot\gamma_4},\phi_{kl}] +\\
&\quad + \frac{1}{2}\varepsilon_{\dot\alpha\dot\gamma_1}\varepsilon_{\dot\gamma_2\dot\gamma_3}\varepsilon^{ij_1j_2k}\delta^{j_4}_{[k}[\tilde\chi_{l]\dot\gamma_4},\phi^{j_3l}]\,.
\end{aligned} \tag{A.11}$$

Putting everything together, the above expansions yield the remaining expansions for the other superfields and moreover show that the superfield equations (A.1) are true order by order in the η-expansion. Thus, the supersymmetric Bogomolny equations (4.25) on $\mathbb{R}^3$ are indeed equivalent to the compatibility conditions (4.42) of the linear system (4.40).

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B. Transversal gauge

The general case. In (7.25), we noticed a freedom in splitting the Lie-algebra valued function ϕ_{+-} defined in (7.22) according to

$$\phi_{+-} = \phi_+ - \phi_- = (\phi_+ + \omega) - (\phi_- + \omega) \ =: \ \tilde{\phi}_+ - \tilde{\phi}_- \,. \tag{B.1}$$

This freedom can be used to guarantee that the infinitesimal deformations of the gauge potential obtained by

$$\delta\hat{\mathcal{A}}^+_{\dot{\alpha}} = \bar{\nabla}^+_{\dot{\alpha}}\phi_\pm \qquad \text{and} \qquad \delta\hat{\mathcal{A}}^i_+ = \bar{\nabla}^i_+\phi_\pm \,, \tag{B.2}$$

are in transversal gauge, i.e., the condition $\eta^{\dot{\alpha}}_i\delta\mathcal{A}^i_{\dot{\alpha}} = 0$ is satisfied, cf. (4.44). Upon expanding the functions $\phi_\pm$ in powers of $\lambda_\pm$ in their respective domains, i.e.

$$\phi_\pm = \sum_{n=0}^{\infty} \lambda_+^{\pm n}\phi_\pm^{(n)} \,, \tag{B.3}$$

we deduce from (B.2)

$$\begin{aligned} \delta\mathcal{A}^i_{\dot{1}} \ &= D^i_{\dot{1}}\phi_+^{(0)} - D^i_{\dot{2}}\phi_+^{(1)} = D^i_{\dot{1}}\phi_-^{(0)} \,, \\ \delta\mathcal{A}^i_{\dot{2}} \ &= D^i_{\dot{2}}\phi_+^{(0)} = -D^i_{\dot{1}}\phi_-^{(1)} + D^i_{\dot{2}}\phi_-^{(0)} \,. \end{aligned} \tag{B.4}$$

The contraction of these equations with $\eta^{\dot{\alpha}}_i$ yields the constraints

$$\mathcal{D}\phi_-^{(0)} + \mathcal{D}\omega = \eta^{\dot{2}}_i D^i_{\dot{1}}\phi_-^{(1)} \qquad \text{and} \qquad \mathcal{D}\phi_+^{(0)} + \mathcal{D}\omega = \eta^{\dot{1}}_i D^i_{\dot{2}}\phi_+^{(1)} \,, \tag{B.5}$$

where we have used the fact that $\tilde{\phi}_\pm^{(0)} = \phi_\pm^{(0)} + \omega$ and $\tilde{\phi}_\pm^{(1)} = \phi_\pm^{(1)}$, respectively. Thus, a splitting (B.1) with an ω satisfying (B.5) yields a deformation of the super gauge potential which respects the transversal gauge condition.

The example. Let us briefly comment on the transversal gauge condition discussed in the previous paragraph in the case of the example presented in section 7.2. Equation (B.5) simplifies in our case (7.31) to

$$\mathcal{D}\phi_-^{(0)} + \mathcal{D}\omega = \eta^{\dot{2}}_i \partial^i_{\dot{1}}\phi_-^{(1)} \qquad \text{and} \qquad \mathcal{D}\phi_+^{(0)} + \mathcal{D}\omega = \eta^{\dot{1}}_i \partial^i_{\dot{2}}\phi_+^{(1)} \,. \tag{B.6}$$

Since our particular deformation (7.30) is of fourth order in the fermionic coordinates, we may assume that $\omega = \eta^{\dot{\gamma}_1\cdots\dot{\gamma}_4}\omega_{\dot{\gamma}_1\cdots\dot{\gamma}_4}$. Then, after some algebraic manipulations, the expansions of $\phi_\pm^{(n)}$ (see e.g. (7.37)) together with (B.6) and the ansatz for ω lead to

$$\omega = -\eta^{\dot{2}\dot{2}\dot{2}\dot{1}}\overset{\circ}{\phi}{}_-^{0(3)} - 3\eta^{\dot{2}\dot{2}\dot{1}\dot{1}}\overset{\circ}{\phi}{}_-^{0(2)} - 3\eta^{\dot{2}\dot{1}\dot{1}\dot{1}}\overset{\circ}{\phi}{}_-^{0(1)} - \eta^{\dot{1}\dot{1}\dot{1}\dot{1}}\overset{\circ}{\phi}{}_-^{0(0)} \,, \tag{B.7}$$

i.e., this particular choice of ω ensures the preservation of transversal gauge for the perturbation (7.30).

C. Signature $(++-)$

In section 2.1, we defined the twistor space of a real four-dimensional manifold X as the bundle (2.1) of almost complex structures compatible with a metric g, which yielded (2.10) as the twistor space of the euclidean space $(\mathbb{R}^4, \delta_{\mu\nu})$. In fact, in euclidean signature, one can define the twistor space of X in three equivalent ways, the first being the above one.

Two further definitions of the twistor space. The second definition assumes that X admits a spin structure. Then one can introduce the vector bundle

$$\mathcal{S} := P(X, \mathrm{Spin}(4)) \times_{\mathrm{Spin}(4)} \mathbb{C}^4 \tag{C.1}$$

of Dirac spinors on X. Since $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, this bundle decomposes into a direct sum $\mathcal{S} = \mathcal{S}_L \oplus \mathcal{S}_R$ of bundles of left- and right-handed Weyl spinors. Using the latter bundle, one can define the twistor space of X as the projectivization [23]

$$\mathcal{Z} := P(\mathcal{S}_R) \tag{C.2}$$

of the bundle $\mathcal{S}_R \to X$ and $\mathcal{Z}$ has again projective lines $\mathbb{C}P^1 \cong S^2$ as fibres. Note that in this definition, the spin structure on X was only needed for introducing the bundle $\mathcal{S}_R \to X$ but not for its projectivization $P(\mathcal{S}_R)$.

The third definition is obtained by considering the vector bundle $\Lambda^2 T^* X$ of two-forms on X. Using the Hodge star operator, one can split $\Lambda^2 T^* X$ into the direct sum $\Lambda^2 T^* X = \Lambda^2_+ T^* X \oplus \Lambda^2_- T^* X$ of the subbundles of self-dual and anti-self-dual two-forms on X. Then the twistor space $\mathcal{Z}$ of X can be defined as the unit sphere bundle

$$\mathcal{Z} := S(\Lambda^2_- T^* X) \tag{C.3}$$

in the vector bundle $\Lambda^2_- T^* X$.

Kleinian case. Although these definitions are all equivalent in the euclidean case, only the latter two are equivalent in the kleinian case of signature $(++--)$ and they differ from definition (2.1). In particular, for the space $\mathbb{R}^{2,2} := (\mathbb{R}^4, g_{\mu\nu})$ with $(g_{\mu\nu}) = \mathrm{diag}(+1,+1,-1,-1)$ we obtain both from (C.2) and (C.3) the space

$$\mathcal{Z} \cong \mathbb{R}^{2,2} \times \mathbb{C}P^1 . \tag{C.4}$$

On the other hand, the definition (2.1) from section 2 yields the space

$$\tilde{\mathcal{Z}} \cong \mathbb{R}^{2,2} \times H^2 = (\mathbb{R}^{2,2} \times H^2_+) \cup (\mathbb{R}^{2,2} \times H^2_-) =: \mathcal{Z}_+ \cup \mathcal{Z}_- , \tag{C.5}$$

which is an open subset of $\mathcal{Z}$. Here, $H^2 = H^2_+ \cup H^2_-$ is the two-sheeted hyperboloid and $H^2_\pm = \{\lambda_\pm \in U_\pm \,|\, |\lambda_\pm| < 1\} \cong \mathrm{SU}(1,1)/\,\mathrm{U}(1)$ are open discs. In fact,

$$\mathcal{Z} = \mathcal{Z}_+ \cup \mathcal{Z}_0 \cup \mathcal{Z}_- , \tag{C.6}$$

where $\mathcal{Z}_0 \cong \mathbb{R}^{2,2} \times S^1$ is a boundary for both $\mathcal{Z}_+$ and $\mathcal{Z}_-$.

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The twistor space of $\mathbb{R}^{2,2}$ is again the space (2.10), which can be written as the union

$$\mathcal{P}^3 = \mathcal{O}(1) \oplus \mathcal{O}(1) = \mathcal{P}^3_+ \cup \mathcal{P}_0 \cup \mathcal{P}^3_- = \tilde{\mathcal{P}}^3 \cup \mathcal{P}_0 \tag{C.7}$$

of three disjoint domains, $\mathcal{P}^3_\pm = \mathcal{P}^3|_{H^2_\pm}$ and $\mathcal{P}_0 = \mathcal{P}^3|_{S^1}$ since $\mathbb{C}P^1 = H^2_+ \cup S^1 \cup H^2_-$. There is a natural map from $\mathcal{Z}$ into $\mathcal{P}^3$ which is a real-analytic bijection between $\mathcal{Z}_\pm$ and $\mathcal{P}^3_\pm$,

$$\tilde{\mathcal{P}}^3 \;\cong\; \tilde{\mathcal{Z}}\,, \tag{C.8}$$

but this map becomes the real fibration $\mathcal{Z}_0 \to \mathcal{T}^3 \subset \mathcal{P}_0$ over a real three-dimensional submanifold $\mathcal{T}^3$ of $\mathcal{P}_0$ (see e.g. [47, 48]).

Reduction to mini-twistor spaces. This directly carries over to the mini-twistor space, which is again obtained by taking the quotient of the twistor space with respect to the abelian group $\mathscr{G}_\mathbb{C} \cong \mathbb{C}$ defined in section 3.2. From (C.7), we thus have $\mathcal{P}^2 := \mathcal{P}^3/\mathscr{G}_\mathbb{C} \cong \mathcal{O}(2)$ while (C.8) yields an open subset $\tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}^3/\mathscr{G}_\mathbb{C}$ of $\mathcal{P}^2$. These considerations readily generalize to the supertwistor spaces and we obtain

$$\mathcal{P}^{2|4} = \mathcal{P}^{3|4}/\mathscr{G}_\mathbb{C} \tag{C.9}$$

and an open subset

$$\tilde{\mathcal{P}}^{2|4} = \tilde{\mathcal{P}}^{3|4}/\mathscr{G}_\mathbb{C} \tag{C.10}$$

in $\mathcal{P}^{2|4}$ together with the open subset of the CR supertwistor space,

$$\tilde{\mathcal{F}}^{5|8} \;\cong\; \mathbb{R}^{2,2|8} \times H^2 \;\subset\; \mathcal{F}^{5|8} \;\;\cong\; \mathbb{R}^{2,2|8} \times S^2\,. \tag{C.11}$$

Modifications in signature $(++-)$**.** In fact, all of these spaces appear in the twistor correspondence between hBF theory, phCS theory and a supersymmetric Bogomolny model on the space $\mathbb{R}^{2,1} = (\mathbb{R}^3, g)$ with the metric $g = \mathrm{diag}(+1,+1,-1)$. Namely, one uses the spaces $\tilde{\mathcal{P}}^{2|4}$ and $\tilde{\mathcal{F}}^{5|8}$ in the Dolbeault description of these correspondences and the spaces $\mathcal{P}^{2|4}$ and $\mathcal{F}^{5|8}$ in the Čech description via transition functions. There are only minor modifications to be made to all the formulæ of the euclidean case to hold also here. First, one replaces the reality condition (2.16) and (2.33) by[32]

$$x^{2\dot{2}} = \bar{x}^{1\dot{1}} = -\mathrm{i}(x^1 - \mathrm{i}x^2)\,, \qquad x^{2\dot{1}} = \bar{x}^{1\dot{2}} = -\mathrm{i}(x^4 - \mathrm{i}x^3) \qquad \text{and} \qquad \eta_i^{\dot{2}} = \bar{\eta}_i^{\dot{1}}\,, \tag{C.12}$$

which corresponds together with (3.11) to the anti-linear involution

$$\hat{\tau}(w^1_\pm, w^2_\pm, \eta_i^\pm) = \left(\frac{\bar{w}^1_\pm}{(\bar{w}^2_\pm)^2}, \frac{1}{\bar{w}^2_\pm}, \frac{1}{\bar{w}^2_\pm}\bar{\eta}_i^\pm\right) \tag{C.13}$$

on the mini-supertwistor space. Second, the hBF theory is considered on the supermanifold $\tilde{\mathcal{P}}^{2|4} \subset \mathcal{P}^{2|4}$ and the phCS theory on the open subset $\tilde{\mathcal{F}}^{5|8}$ of the CR supertwistor space $\mathcal{F}^{5|8}$. Thus, one substitutes the space $\mathbb{C}P^1$ by the two-sheeted hyperboloid $H^2 = \mathbb{C}P^1 \backslash S^1 =$

[32]See e.g. [3] for more details.

$H^2_+ \cup H^2_-$ and uses

$$(\hat{\lambda}^+_{\dot{\alpha}}) = \begin{pmatrix} \bar{\lambda}_+ \\ 1 \end{pmatrix}, \qquad (\hat{\lambda}^-_{\dot{\alpha}}) = \begin{pmatrix} 1 \\ \bar{\lambda}_- \end{pmatrix}, \qquad \hat{\lambda}^{\dot{\alpha}}_{\pm} = \varepsilon^{\dot{\alpha}\dot{\beta}} \hat{\lambda}^{\pm}_{\dot{\beta}}\,,$$
$$\gamma_{\pm} = \pm \frac{1}{1 - \lambda_{\pm}\bar{\lambda}_{\pm}}\,, \qquad \lambda_{\pm} \in H^2_{\pm} \tag{C.14}$$

instead of $\hat{\lambda}^{\pm}_{\dot{\alpha}}$, $\hat{\lambda}^{\dot{\alpha}}_{\pm}$ and $\gamma_{\pm}$ as given in (2.25), (3.35) and (2.22). All other formulæ including the equations of motion for phCS theory on $\tilde{\mathcal{F}}^{5|8}$ and the field expansions of $\mathcal{A}_{\mathcal{T}}$ keep their form. The resulting Bogomolny-type field equations on $\mathbb{R}^{2,1}$ will only differ by some signs in front of the interaction terms. All this also holds for the M-deformed case, which eventually involves the spaces $\tilde{\mathcal{P}}^{2|4}_M$ and $\tilde{\mathcal{F}}^{5|8}_M$.

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